# MODEL FROBENIOIDS

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The following lecture notes were prepared for the IUT Theory of Shinichi Mochizuki conference to be held Monday, December 7, 2015 to Friday, December 11, 2015 at the Mathematical Institute at the University of Oxford and sponsored by the Clay Mathematics Institute [1] and organized by Ivan Fesenko. They are based entirely on work of Mochizuki himself: [3] and [4], available on the Mochizuki's homepage. These notes are still very rough and may contain mistakes. This is the first talk on Frobenioids, the second is to be given by Weronika Czerniawska.

## 1. An example of a model Frobeniod of geometric origin

This is Example 6.1 of [3]. Let $V$ be a proper, normal variety over $k$, $K$ the function field of $V$ and $\tilde{K}$ a Galois extension of $k$ (i.e. algebraic and the fixed field of $Aut(\tilde{K}/K)$ is $K$). Let $G$ be the Galois group of $\tilde{K}$ over $K$. The group $G$ has a *profinite topology*: a fundamental system of neighborhoods of the identity is given by the Galois groups of $\tilde{K}$ over subfields of $\tilde{K}$ which are finite etensions of $K$. Let $\mathbb{D}_K$ be a set of $\mathbb{Q}$-Cartier prime divisors on $V$.

The *Galois category* $\mathcal{B}(G)$ has objects finite sets equipped with continuous $G$-action and morphisms are morphisms of $G$-sets. The *connected objects* of $\mathcal{B}(G)$ are those finite sets with continuous $G$-action which do not break up into a disjoint union of non-empty $G$-sets. If we think of $Spec(\tilde{K})$ as a $G$-bundle on $Spec(K)$, any object $S$ of $\mathcal{B}(G)$ defines a scheme $Spec(L_S)$ where $K \subset L_S \subset \tilde{K}$ and $Spec(L_S) = (Spec(\tilde{K}) \times S)/G$ or $L_S = (K[S] \otimes_K \tilde{K})^G$. Fix a collection $\mathbb{D}_K$ of $\mathbb{Q}$-Cartier prime divisors on $V$. Let $V[L]$ be the normalization of $V$ in $L$. Write $\mathbb{D}_L$ for the set of prime divisors of $V[L]$ that map into (subvarieties of) prime divisors of $\mathbb{D}_K$. Assume these are all $\mathbb{Q}$-Cartier. Define a functor from $\mathcal{B}(G)^0$ to the category of commutative monoids

$$\Phi : \mathcal{B}(G)^0 \to \mathfrak{Mon}$$

by letting $\Phi(L)$ be the monoid of effective Cartier divisors $D$ on $V[L]$ with support in $\mathbb{D}_L$. Then the groupification of $\Phi(L)$ is a subgroup of the group of Cartier divisors on $V[L]$. This groupification gives another functor $\Phi^{gp} : \mathcal{B}(G)^0 \to \mathfrak{Mon}$. We can define a functor into commutative groups (thought of as monoids also)

$$\mathbb{B} : \mathcal{B}(G)^0 \to \mathfrak{Mon}$$

by letting $\mathbb{B}(L)$ be the group of rational functions $f$ such that every prime divisor at which $f$ has a zero or pole belongs to $\mathbb{D}_L$. There is a natural (functorial in $L$) morphism

$$\mathbb{B}(L) \to \Phi^{gp}(L)$$

simply using the usual divisor of zeros minus divisor of poles construction on $V[L]$. Here, we are interpresting $\mathbb{B}(L)$ as rational functions on $V[L]$. To this data, we can associate a category whose objects are pairs $(L, \mathcal{L})$ where $L \in \mathcal{B}(G)^0$ and $\mathcal{L}$ is a line bundle on $V[L]$

representable by a Cartier divisor with support of $\mathbb{D}_L$. Given a pair of objects $(L, \mathcal{L})$ and $(M, \mathcal{M})$ a morphism consists of

(1) a morphism $Spec(L) \to Spec(M)$ over $Spec(K)$ (inducing a morphism $V[L] \to V[M]$ over $V$).
(2) an element $d \in \mathbb{N}_{\geq 1}$
(3) a morphism of line bundles $\mathcal{L}^{\otimes d} \to \mathcal{M}|_{V[L]}$ whose zero locus is a Cartier divisor supported in $\mathbb{D}_L$.

We now pass to a more general language to understand why this is a category and how to work with it.

## 2. Commutative Monoids and elementary Frobenioids

In this section we give basic definitions concerning monoids on categories and define the elementary Frobenioid $\mathbb{F}_\Phi$. We write the operation on monoids additively. Given a monioid we let $M^{\pm}$ be the submonoid of invertible elements and $M^{char} = M/M^{\pm}$. The groupification of a monoid is called $M^{gp}$. The category of commutative monoids is denoted $\mathfrak{Mon}$. The next two definitions list some properties that monoids can have.

**Definition 2.1.** A monoid is called

(1) *sharp* if its only invertible element is 0
(2) *integral* if the natural map of monoids $M \to M^{gp}$ from a moidoid into its groupification is injective
(3) *saturated* if any $a \in M^{gp}$ for which $na$ is in the image of $M$ for some $n \in \mathbb{N}_{\geq 1}$ is in the image of $M$.
(4) *of characteristic type* if the fibers of $M \to M^{char}$ are torsors over $M^{\pm}$
(5) *group-like* if $M^{char}$ is zero.

**Definition 2.2.** A monoid is called

(1) *pre-divisorial* if it is integral, saturated and of characteristic type
(2) *divisorial* if it is pre-divisoral and sharp.

**Definition 2.3.** A morphism $\phi : M_1 \to M_2$ in $\mathfrak{Mon}$ is called *characteristically injective* if the induced morphism $M_1^{char} \to M_2^{char}$ is injective.

**Definition 2.4.** A category is called *connective* if its associated graph (built from associating verticies and edges to objects and morphisms of the category) is connected. A category is called *totally epimorphic* if every morphism of the category is an epimorphism.

**Definition 2.5.** A morphism $\beta : B \to A$ in a category $\mathcal{C}$ is called

(1) *fiberwise-surjective* if for every arrow $\gamma : C \to A$ of $\mathcal{C}$ there exist arrows $\delta_B : D \to B$ and $\delta_C : D \to C$ such that $\beta \circ \delta_B = \gamma \circ \delta_C$
(2) a *FSM* (fiberwise-surjective monomorphism) if it is fiberwise-surjective and a monomorphism

**Definition 2.6.** A monoid on a category $\mathcal{D}$ is a contravariant functor $\Phi : \mathcal{D} \to \mathfrak{Mon}$ such that

- each of the morphisms $\alpha^* : \Phi(A) \to \Phi(B)$ is characteristically injective

- if $\alpha$ is a FSM then $\alpha^*$ is an isomorphism

where we have used the notation $\Phi(\alpha : B \to A) = \alpha^* : \Phi(A) \to \Phi(B)$.

We say that a monoid on a category $\Phi : \mathcal{D} \to \mathfrak{Mon}$ has certain property when for each object $d$ of $\mathcal{D}$, $\Phi(d)$ has that property.

**Definition 2.7.** If $\Phi$ is a monoid on a category $\mathcal{D}$ then the *elementary Frobenioid associated to* $\Phi$ is a certain category $\mathbb{F}_\Phi$ with the same objects as $\mathcal{D}$: $ob(\mathbb{F}_\Phi) = ob(\mathcal{D})$. If $A$ is an object in $\mathbb{F}_\Phi$ then we let $A_\mathcal{D}$ be the associated object in $\mathcal{D}$. The set of morphisms $\phi : A \to B$ in $\mathbb{F}_\Phi$ is defined to be the set of triples

$$\{(\phi_\mathcal{D}, Z_\phi, n_\phi) \ | \ \phi_\mathcal{D} \in \mathrm{Hom}_\mathcal{D}(A_\mathcal{D}, B_\mathcal{D}), Z_\phi \in \Phi(A_\mathcal{D}), n_\phi \in \mathbb{N}_{\geq 1}\}.$$

We sometimes write $\mathrm{Base}(A) = A_\mathcal{D}$ and $\mathrm{Base}(\phi) = \phi_\mathcal{D}$. We refer to $n_\phi$ as the Frobenius degree $\deg_{Fr}(\phi)$. We refer to $Z_\phi$ as the zero divisor $\mathrm{Div}(\phi)$ of $\phi$. Given $\phi : A \to B$ and $\psi : B \to C$ in $\Phi$ their composition in $\mathbb{F}_\Phi$ is defined to be

$$\psi \circ \phi = (\psi_\mathcal{D} \circ \phi_\mathcal{D}, \phi^*_\mathcal{D}(Z_\psi) + n_\psi \cdot Z_\phi, n_\psi \cdot n_\phi) : A \to C.$$

**Example 2.8.** If $\mathcal{D}$ is the category with one object and one morphism, then a functor $\Phi$ is determined by a commutative monoid $M$. This functor is called $\Phi_M$. The associated *elemetary Frobenioid associated to* $M$ is called $\mathbb{F}_{\Phi_M}$ has just one object. It is determined by the endomorphism monoid (called $\mathbb{F}_M$) of its one object. This monoid is just the set $M \times \mathbb{N}_{\geq 1}$ wquipped with the monoid structure

$$(a_1, n_1) \cdot (a_2, n_2) = (a_1 + n_1 \cdot a_2, n_1 \cdot n_2).$$

There is a short exact sequence of monoids

$$0 \to M \to \mathbb{F}_M \to \mathbb{N}_{\geq 1} \to \{1\}.$$

The *standard Frobenioid* $\mathbb{F}_{\mathbb{Z}_{\geq 0}}$ is written simply $\mathbb{F}$.

**Definition 2.9.** Let $\Phi$ be a divisorial monoid on a connected, totally epimorphic category $\mathcal{D}$. Let $\mathcal{C}$ be a connected, totally epimorphic category. A *pre-Frobenioid structure* on a $\mathcal{C}$ is a covariant functor $\mathcal{C} \to \mathbb{F}_\Phi$. Such a category together with this type of functor is called a *pre-Frobenioid.*

**Definition 2.10.** Let $\mathcal{C} \to \mathbb{F}_\Phi$ be a pre-Frobenioid. Let $\phi$ be a morphism in $\mathcal{C}$. We say that $\phi$ is *linear* if $\deg_{Fr}(\phi) = 1$ and that $\phi$ is *isometric* if $\mathrm{Div}(\phi) = 0$. We say that $\phi$ is a *base-identity* morphism if $\mathrm{Base}(\phi)$ is the identity. We say that $\phi$ is a *pre-step* if it is linear and $\mathrm{Base}(\phi)$ is an isomorphism. We call $\phi$ a *pull-back* morphism if the natural transformation of contravariant functors on $\mathcal{C}$

$$Hom_\mathcal{C}(-, A) \to Hom_\mathcal{C}(-, B) \times_{Hom_\mathcal{D}(-,B_\mathcal{D})|_\mathcal{C}} Hom_\mathcal{D}(-, A_\mathcal{D})|_\mathcal{C}$$

induced by $\phi$ is a natural equivalence, where $|_\mathcal{C}$ denotes restriction to $\mathcal{C}$.

The following table sumarizes some known facts about types of morphisms:

| type of morphism | projection to base | zero divisor | Frobenius degree |
|---|---|---|---|
| linear | ? | ? | 1 |
| isometric | ? | 0 | ? |
| base-identity | identity | ? | ? |
| base-isomorphism | isomorphism | ? | ? |
| pre-step | isomorphism | ? | 1 |
| pull-back | ? | 0 | 1 |

## 3. model Frobenioids

The purpose of this section is to introduce model Frobenioids, an explicit type of category which will be denoted $\underline{\mathcal{C}}$. The ingredients that are needed to define a model Frobenioid are as follows:

(1) $\mathcal{D}$ is a connected, totally epimorphic category.
(2) $\underline{\Phi} : \mathcal{D} \to \mathfrak{Mon}$ be a divisorial monoid on $\mathcal{D}$
(3) $\mathbb{B} : \mathcal{D} \to \mathfrak{Mon}$ a group-like monoid on $\mathcal{D}$
(4) $\mathrm{Div}_{\mathbb{B}} : \mathbb{B} \to \underline{\Phi}^{gp}$ a homomorphism of monoids on $\mathcal{D}$.

The image of $\mathrm{Div}_{\mathbb{B}}$ is a group-like monoid denoted by $\underline{\Phi}^{birat} \subset \underline{\Phi}^{gp}$. It is not difficult to check that the following construction defines a category $\underline{\mathcal{C}}$.

**Definition 3.1.** The objects of $\underline{\mathcal{C}}$ are pairs $(A_{\mathcal{D}}, \alpha)$ where $A_{\mathcal{D}}$ is an object of $\mathcal{D}$ and $\alpha \in \underline{\Phi}(A_{\mathcal{D}})^{gp}$. A morphism $\phi$ between $A = (A_{\mathcal{D}}, \alpha)$ and $B = (B_{\mathcal{D}}, \beta)$ is a quadruple

$$\phi = (\deg_{Fr}(\phi), \mathrm{Base}(\phi), \mathrm{Div}(\phi), u_\phi)$$

where $\deg_{Fr}(\phi) \in \mathbb{N}_{\geq 1}$, $\mathrm{Base}(\phi) : A_{\mathcal{D}} \to B_{\mathcal{D}}$ in $\mathcal{D}$, $\mathrm{Div}(\phi) \in \underline{\Phi}(A_{\mathcal{D}})$ and $u_\phi \in \mathbb{B}(A_{\mathcal{D}})$ whose image $\mathrm{Div}_{\mathbb{B}}(u_\phi) \in \underline{\Phi}(A_{\mathcal{D}})^{gp}$ satisfies

$$\deg_{Fr}(\phi) \cdot \alpha + \mathrm{Div}(\phi) = (\underline{\Phi}^{gp}(\mathrm{Base}(\phi)))(\beta) + \mathrm{Div}_{\mathbb{B}}(u_\phi).$$

The composition in this category is defined by

$$\deg_{Fr}(\psi \circ \phi) = \deg_{Fr}(\psi) \deg_{Fr}(\phi)$$

$$\mathrm{Base}(\psi \circ \phi) = \mathrm{Base}(\psi) \circ \mathrm{Base}(\phi)$$

$$\mathrm{Div}(\psi \circ \phi) = (\underline{\Phi}(\mathrm{Base}(\phi)))(\mathrm{Div}(\psi)) + \deg_{Fr}(\phi) \cdot \mathrm{Div}(\phi)$$

and

$$u_{\psi \circ \phi} = \mathbb{B}(\mathrm{Base}(\phi)))(u_\psi) + \deg_{Fr}(\psi) \cdot u_\phi.$$

There is a natural functor

$$\underline{\mathcal{C}} \to \mathbb{F}_{\underline{\Phi}}$$

given by

$$(A_{\mathcal{D}}, \alpha) \mapsto A_{\mathcal{D}}$$

and

$$\phi = (\deg_{Fr}(\phi), \mathrm{Base}(\phi), \mathrm{Div}(\phi), u_\phi) \mapsto (\phi_{\mathcal{D}} = \mathrm{Base}(\phi), Z_\phi = \mathrm{Div}(\phi), n_\phi = \deg_{Fr}(\phi)).$$

This shows that a model Frobenioid is a special type of pre-Frobenioid. Professor Mochizuki also introduced in [4] a torsor theoretic approach to model Frobenioids. We now outline this and explain why it is equivalent to the other approach. Let $\underline{\Phi} : \mathcal{D} \to \mathfrak{Mon}$ be a divisorial monoid on $\mathcal{D}$, $\mathbb{B} : \mathcal{D} \to \mathfrak{Mon}$ a group-like monoid on $\mathcal{D}$, $\mathrm{Div}_{\mathbb{B}} : \mathbb{B} \to \underline{\Phi}^{gp}$ a homomorphism of monoids on $\mathcal{D}$. We define a category to be called $\underline{\mathcal{C}}^{tor}$. An object is a triple $(A_{\mathcal{D}}, T_A, \tau_A)$ where $A_D$ is an object of $\mathcal{D}$, $T_A$ is a $\mathbb{B}(A_{\mathcal{D}})$-torsor, $\tau_A$ is a trivialization of the associated $\underline{\Phi}(A_{\mathcal{D}})^{gp}$-torsor induded from $T_A$ by the homomorphism $\mathrm{Div}_{\mathbb{B}}(A_{\mathcal{D}}) : \mathbb{B}(A_{\mathcal{D}}) \to \underline{\Phi}(A_{\mathcal{D}})^{gp}$. A morphism between $A = (A_{\mathcal{D}}, T_A, \tau_A)$ and $B = (B_{\mathcal{D}}, T_B, \tau_B)$ is a triple

$$\phi = (d_\phi, \mathrm{Base}(\phi), f_\phi)$$

where $d_\phi \in \mathbb{N}_{\geq 1}$, $\mathrm{Base}(\phi) : A_{\mathcal{D}} \to B_{\mathcal{D}}$ is a morphism in $\mathcal{D}$. The last component, $f_\phi$ is an isomorphism of $\mathbb{B}(A_{\mathcal{D}})$-torsors $T_A^{\otimes d_\phi} \to \phi^* T_B$. Here $T_A^{\otimes d_\phi}$ is the $\mathbb{B}(A_{\mathcal{D}})$-torsor induced from $T_A$ be the homomorphism $\mathbb{B}(A_{\mathcal{D}}) \to \mathbb{B}(A_{\mathcal{D}})$ given by multiplication by $d_\phi$ and $\phi^* T_B$ is the $\mathbb{B}(A_{\mathcal{D}})$-torsor induced the $\mathbb{B}(B_{\mathcal{D}})$-torsor $T_B$ via the morphism $\mathbb{B}(\phi) : \mathbb{B}(B_{\mathcal{D}}) \to \mathbb{B}(A_{\mathcal{D}})$. We require that $f_\phi$ maps the trivialization $\tau_A^{\otimes d_\phi}$ to an element in the $\underline{\Phi}(A_{\mathcal{D}})$-orbit of $\phi^* \tau_B$. Given another such morphism $\psi = (d_\psi, \mathrm{Base}(\psi), f_\psi)$ between $B = (B_{\mathcal{D}}, T_B, \tau_B)$ and $C = (C_{\mathcal{D}}, T_C, \tau_C)$, their composition is made up of $d_{\psi\circ\phi} = d_\psi d_\phi$, obviously $\mathrm{Base}(\psi \circ \phi) = \mathrm{Base}(\psi) \circ \mathrm{Base}(\phi)$ and finally

$$T_A^{\otimes(d_\psi d_\phi)} \cong (T_A^{\otimes d_\phi})^{\otimes d_\psi} \to (\phi^* T_B)^{\otimes d_\psi} \cong \phi^*(T_B^{\otimes d_\psi}) \to \phi^*(\psi^* T_C)$$

gives $f_{\psi\circ\phi} : T_A^{\otimes d_{\psi\circ\phi}} \to (\psi \circ \phi)^* T_C$. In other words, $f_{\psi\circ\phi} = (\phi^* f_\psi) \circ f_\phi^{d_\psi}$.

**Lemma 3.2.** *These two notions of Frobenioids agree.*

*Proof.* We define an equivalence of categories $\underline{\mathcal{C}} \to \underline{\mathcal{C}}^{tor}$. An object $(A_{\mathcal{D}}, \alpha)$ is mapped to the object $(A_{\mathcal{D}}, T_A, \tau_A)$ where $T_A$ is the trivial $\mathbb{B}(A_{\mathcal{D}})$-torsor and $\tau_A$ is the trivialization of the trivial $\Phi(A_{\mathcal{D}})^{gp}$-torsor given by $\alpha \in \Phi(A_{\mathcal{D}})^{gp}$. Suppose that we are given a morphism

$$\phi = (\deg_{Fr}(\phi), \mathrm{Base}(\phi), \mathrm{Div}(\phi), u_\phi)$$

between $(A_{\mathcal{D}}, \alpha)$ and $(B_{\mathcal{D}}, \beta)$. Recall that $u_\phi \in \mathbb{B}(A_{\mathcal{D}})$ and $\mathrm{Div}(\phi) \in \underline{\Phi}(A_{\mathcal{D}})$ and we have

$$\deg_{Fr}(\phi) \cdot \alpha + \mathrm{Div}(\phi) = (\underline{\Phi}^{gp}(\mathrm{Base}(\phi)))(\beta) + \mathrm{Div}_{\mathbb{B}}(u_\phi) = \phi^*\beta + \mathrm{Div}_{\mathbb{B}}(u_\phi). \tag{3.1}$$

We send it to the morphism $(d_\phi, \mathrm{Base}(\phi), f_\phi)$ where $d_\phi = \deg_{Fr}(\phi)$ and

$$f_\phi : T_A^{\otimes d_\phi} \to \phi^* T_B$$

which should map an element in the $\underline{\Phi}(A_{\mathcal{D}})$-orbit of $d_\phi \cdot \alpha$ to $\phi^*\beta$ [1]. Every object in $\underline{\mathcal{C}}^{tor}$ is isomorphic to something of this form. Indeed, given an object $(A_{\mathcal{D}}, T_A, \tau_A)$ chose any trivialization of $\mu : \mathbb{B}(A_{\mathcal{D}}) \to T_A$ of $T_A$. The difference of $\tau_A$ with the associated trivialization to $\mu$ of the associated $\underline{\Phi}(A_{\mathcal{D}})^{gp}$-torsor $S_A$ induced from $T_A$ by the homomorphism $\mathrm{Div}_{\mathbb{B}}(A_{\mathcal{D}}) : \mathbb{B}(A_{\mathcal{D}}) \to \underline{\Phi}(A_{\mathcal{D}})^{gp}$ is an element $\alpha$ of $\Phi(A_{\mathcal{D}})^{gp}$. Now the image of $(A_{\mathcal{D}}, \alpha)$ looks like $(A_{\mathcal{D}}, \mathbb{B}(A_{\mathcal{D}}), \alpha)$. We have an isomorphism

$$(1, \mathrm{id}, \mu) : (A_{\mathcal{D}}, \mathbb{B}(A_{\mathcal{D}}), \alpha) \to (A_{\mathcal{D}}, T_A, \tau_A)$$

and $\underline{\Phi}^{gp}(A_{\mathcal{D}}) \overset{\tau_A}{\to} S_A \overset{\mu^{-1}}{\to} \underline{\Phi}^{gp}(A_{\mathcal{D}})$ takes the unit to $\alpha$. Therefore, the functor is essentially surjective. The fully faithfulness of the functor follows simply because given a morphim $\phi^t$

[1] Mochizuki wrote instead of this that [4] it should map $d_\phi \cdot \alpha$ to an element in the $\underline{\Phi}(A_{\mathcal{D}})$-orbit of $\phi^*\beta$ but when comparing with the other definition of model Frobenioids the version I wrote in the main text made more sense to me. This is probably a minor issue.

from $\underline{\mathcal{C}}^{tor}$ and using equation (3.1), we can recover an element $\mathrm{Div}(\phi)$ and uniquely define a morphism $\phi$ in $\underline{\mathcal{C}}$ which maps to $\phi^t$. Consider the morphism

$$x \mapsto x - u_\phi$$

as a morphism of trivial $\mathbb{B}(A_{\mathcal{D}})$-torsors. We have

$$u_{\psi\circ\phi} = \mathbb{B}(\mathrm{Base}(\phi))(u_\psi) + \deg_{Fr}(\psi) \cdot u_\phi$$

which implies that the $f_\phi$ compose correctly.

The associated morphism of trivial $\underline{\Phi}(A_{\mathcal{D}})^{gp}$-torsors is

$$y \mapsto y - \mathrm{Div}_{\mathbb{B}}(u_\phi).$$

This maps the element $\deg_{Fr}(\phi) \cdot \alpha + \mathrm{Div}(\phi)$ which is in the $\underline{\Phi}(A_{\mathcal{D}})$-orbit of $\deg_{Fr}(\phi) \cdot \alpha$ to $\phi^*\beta$. □

## 4. Further Study

The next talk after this one in the conference is to be given by Weronika Czerniawska who has prepared slides (to be available at [1]), including an arithmetic example (Example 6.3 of [3]), p-adic Frobenioids (introduced in The Geometry of Frobenioids II, avalable from Mochizuki's webpage) and more. The most involved type of Frobenioid in IUT is called a tempered Frobenioid which is also a model Frobenioid. A math overflow discussion by Minhyong Kim is available [2]. A *Frobenioid* is a pre-Frobenioid satisfying a fairly intimidating list of conditions based on Definition 2.9 and similar definitions. Elementary Frobenioids and model Frobenioids are particular types of Frobenioids. It seems that to understand IUT one just needs to understand model Frobenioids. However, one could imagine that in the future, more general Frobenioids could be useful.

## References

[1] Ivan Fesenko's webpage `https://www.maths.nottingham.ac.uk/personal/ibf/files/symcor.iut.html`

[2] M. Kim, What is a Frobenioid? `http://mathoverflow.net/questions/195353/what-is-a-frobenioid`

[3] S. Mochizuki, The Geometry of Frobenioids I, available on author's homepage: `http://www.kurims.kyoto-u.ac.jp/~motizuki/top-english.html`.

[4] S. Mochizuki, Responses to questions on Frobenioids, available on author's homepage: `http://www.kurims.kyoto-u.ac.jp/~motizuki/top-english.html`.