On the image of noncommutative local reciprocity map

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0. Introduction

First steps in the direction of an arithmetic noncommutative local class field theory were described in [2] as an attempt to find an arithmetic generalization of the classical abelian class field theory; see [3] for an exposition of its main features. In particular, [2] clarified and simplified the metabelian local class field theory of H. Koch and E. de Shalit [7], [8]. In the noncommutative local class field theory [2] a direct arithmetic description of Galois extensions of a fixed local field F is given by means of noncommutative reciprocity maps between the Galois group $\mathrm{Gal}(L/F)$ of a totally ramified arithmetically profinite Galois extension L/F and a certain subquotient of formal power series in one variable over the algebraic closure of the residue field of F (which, more precisely, is the completion of the maximal unramified extension of the field of norms of L/F). One of the reciprocity maps (see below for definitions) is

$$\mathbf{N}_{L/F}\colon \mathrm{Gal}(L/F) \longrightarrow U^{\diamond}_{\mathcal{N}(L/F)}/U_{N(L/F)}.$$

This map is an injective 1-cocycle (the right hand side has a natural action of the Galois group). It is not surjective, and not a homomorphism in general. In the general case of nonabelian extensions, the description of the Galois group in this approach is given by objects related not only to the ground field F but to L as well.

To describe the image of the reciprocity map one can use a map from $\mathrm{Gal}(L/F)$ to $U^{\diamond}_{\mathcal{N}(L/F)}/Y_{L/F}$ induced by $\mathbf{N}_{L/F}$, where $Y_{L/F}$ is a certain subgroup of $U^{\diamond}_{\mathcal{N}(L/F)}$ containing $U_{N(L/F)}$, such that the induced map is bijective. A key problem is *to obtain as much information as possible about the subgroup* $Y_{L/F}$. Then via the reciprocity map $\mathbf{N}_{L/F}$ this information translates into a description of the Galois group of L/F.

In this short note we suggest a new definition of certain maps f_i (see section 2) for regular extensions L/F. This provides more information on the submodule $Y_{L/F}$.

Needless to say, this arithmetic approach to noncommutative local class field theory is very different from the Langlands approach, which is somehow less arithmetic. From a general point of view it should be quite difficult to get a sufficiently explicit description of $Y_{L/F}$ for an arbitrary class of extensions L/F. There is a nice explicit description in the case of metabelian extensions, see [2],[7],[8]. It is expected there is a good explicit description in the case of p-adic Lie extensions, on the basis of [2] and this work. This may be of use for the local noncommutative Iwasawa theory.

We will assume that the reader has a good knowledge of basic results on local fields, as given for example in [4, Ch.III–IV].

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1. The abelian case: interpretation

We start with a brief description of an interpretation of the abelian reciprocity maps, since it is this interpretation which leads to the construction of noncommutative reciprocity maps.

Let F be a local field with finite residue field whose characteristic is p. Denote by F^{ur} the maximal unramified extension of F in a fixed completion of a separable closure of F and denote by $\mathcal{F}$ be the completion of F^{ur}.

Now we briefly present two (abelian) local reciprocity maps: geometric and arithmetic. Each of them uses the fact that for a finite Galois extension $\mathcal{L}/\mathcal{F}$ the homomorphism

$$\mathrm{Gal}(\mathcal{L}/\mathcal{F}) \longrightarrow \ker N_{\mathcal{L}/\mathcal{F}}/V(\mathcal{L}/\mathcal{F}), \quad \sigma \longmapsto \pi^{\sigma-1}$$

is surjective with the kernel being the derived group of the Galois group. Here $V(\mathcal{L}/\mathcal{F})$ is the augmentation subgroup generated by elements $u^{\sigma-1}$ with $u \in U_{\mathcal{L}}, \sigma \in \mathrm{Gal}(\mathcal{L}/\mathcal{F})$, and π is any prime element of $\mathcal{L}$. For a noncommutative generalization of this, see the first assertion of the Theorem below.

First, we give a description of the geometric reciprocity map. Let $\mathcal{L}/\mathcal{F}$ be a finite Galois extension. Viewing all objects with respect to the pro-algebraic Zariski topology [10] one has a commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(U_{\mathcal{L}}) & \longrightarrow & \overline{U_{\mathcal{L}}} & \xrightarrow{\alpha} & U_{\mathcal{L}} & \longrightarrow & 1 \\
 & & \big\downarrow{\scriptstyle N_{\mathcal{L}/\mathcal{F}}} & & \big\downarrow{\scriptstyle N_{\mathcal{L}/\mathcal{F}}} & & \big\downarrow{\scriptstyle N_{\mathcal{L}/\mathcal{F}}} & & \\
1 & \longrightarrow & \pi_1(U_{\mathcal{F}}) & \longrightarrow & \overline{U_{\mathcal{F}}} & \longrightarrow & U_{\mathcal{F}} & \longrightarrow & 1 \\
 & & & & \big\downarrow & & \big\downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}$$

where $\overline{U_{\mathcal{L}}}$ and $\overline{U_{\mathcal{F}}}$ are the universal covering spaces of $U_{\mathcal{L}}$ and $U_{\mathcal{F}}$. Applying the snake lemma, one has a map

$$\sigma \longmapsto \pi^{\sigma-1} \in U_{\mathcal{L}} \longmapsto N_{\mathcal{L}/\mathcal{F}}\big(\alpha^{-1}(\pi^{\sigma-1})\big) \in \pi_1(U_{\mathcal{F}})/N_{\mathcal{L}/\mathcal{F}}\pi_1(U_{\mathcal{L}})$$

which is the geometric reciprocity homomorphism (this is more or less straightforward from [10]).

For a separable extension L of F put $L^{\mathrm{ur}} = LF^{\mathrm{ur}}$, $\mathcal{L} = L\mathcal{F}$. To define the arithmetic reciprocity map, let L be a finite totally ramified Galois extension of F. Let φ be an element of the absolute Galois group of F such that its restriction to F^{ur} is the Frobenius automorphism of F. Denote by the same notation the continuous extension of φ to the completion of the maximal separable extension of F. Let π be a prime element of L.

There is a commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & U_L & \longrightarrow & U_{\mathcal{L}} & \xrightarrow{1-\varphi} & U_{\mathcal{L}} & \longrightarrow & 1 \\
 & & \big\downarrow{\scriptstyle N_{L/F}} & & \big\downarrow{\scriptstyle N_{\mathcal{L}/\mathcal{F}}} & & \big\downarrow{\scriptstyle N_{\mathcal{L}/\mathcal{F}}} & & \\
1 & \longrightarrow & U_F & \longrightarrow & U_{\mathcal{F}} & \xrightarrow{1-\varphi} & U_{\mathcal{F}} & \longrightarrow & 1 \\
 & & & & \big\downarrow & & \big\downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}$$

and, applying the snake lemma, one has a map

$$\sigma \longmapsto \pi^{\sigma-1} \in U_L \longmapsto N_{\mathcal{L}/\mathcal{F}}\big((1-\varphi)^{-1}(\pi^{\sigma-1})\big) \in U_F/N_{L/F}U_L$$

which is the arithmetic local reciprocity homomorphism [5], [6], [9], [4]. The equation

$$u^{1-\varphi} = \pi^{\sigma-1}$$

plays a fundamental role for the arithmetic reciprocity homomorphism.

Of course, the geometric reciprocity homomorphism can be viewed as the projective limit of the arithmetic reciprocity homomorphisms.

2. The reciprocity map $\mathbf{N}_{L/F}$

We present one of noncommutative reciprocity maps originally defined in [2].

Denote by F^{φ} the fixed subfield of φ in the separable closure of F. Let L/F be a Galois arithmetically profinite extension which is infinite. We will suppose throughout the paper that $L \subset F^{\varphi}$. For the theory of fields of norms of arithmetically profinite extensions see [11] and [4, Ch. III sect. 5]. The field of norms $N(L/F)$ of the extension L/F is a local field of characteristic p with residue field isomorphic to the residue field of F.

Denote by X the norm compatible sequence of prime elements of finite subextensions of F in L which is the part of the unique norm compatible sequence of prime elements in finite extensions of F in F^{φ} (for its existence and uniqueness see the first section of [8]).

Denote by $\mathcal{N}(L/F)$ the completion of the maximal unramified extension of the field $N(L/F)$. Denote by $U^{\diamond}_{\mathcal{N}(L/F)}$ the subgroup of those elements of $U_{\mathcal{N}(L/F)}$ whose $\mathcal{F}$-component belongs to U_F.

The reciprocity map $\mathbf{N}_{L/F}$ is defined as

$$\mathbf{N}_{L/F}\colon \mathrm{Gal}(L/F) \longrightarrow U^{\diamond}_{\mathcal{N}(L/F)}/U_{N(L/F)}, \quad \mathbf{N}_{L/F}(\sigma) = U \bmod U_{N(L/F)},$$

where $U \in U_{\mathcal{N}(L/F)}$ satisfies the (quite similar to the above) equation

$$U^{1-\varphi} = X^{\sigma-1}.$$

It was shown in [2] that the ground $\mathcal{F}$-component of $\mathbf{N}_{L/F}$ equals the arithmetic reciprocity map described above, so $\mathbf{N}_{L/F}$ is indeed a genuine extension of the abelian reciprocity map.

Fix a tower of subfields $F = E_0 - E_1 - E_2 - \dots$, such that $L = \cup E_i$, E_i/F is a Galois extension, and E_i/E_{i-1} is cyclic of degree p for $i > 1$ and E_1/E_0 is cyclic of degree relatively prime to p. Let σ_i be an element of $\mathrm{Gal}(\mathcal{L}/\mathcal{F})$ whose restriction to E_i is a generator of $\mathrm{Gal}(E_i/E_{i-1})$. Denote by $v_{\mathcal{E}_i}$ the discrete valuation of $\mathcal{E}_i = E_i\mathcal{L}$. Put $s_i = v_{\mathcal{E}_i}(\pi_{E_i}^{\sigma_i-1} - 1)$ where π_{E_i} is a prime element of E_i.

The group $U^{\diamond}_{\mathcal{N}(L/F)}$ contains a subgroup $Y_{L/F}$ (which contains $U_{N(L/F)}$) such that the reciprocity map $\mathbf{N}_{L/F}$ induces a bijection between $\mathrm{Gal}(L/F)$ and $U^{\diamond}_{\mathcal{N}(L/F)}/Y_{L/F}$, see [2, Th.2]. To get more information on $Y_{L/F}$ and its more explicit description, [2] uses certain liftings

$$f_i\colon U_{\mathcal{E}_i}^{\sigma_i-1} \longrightarrow U_{\mathcal{N}(L/E_i)} \longrightarrow U_{\mathcal{N}(L/F)}.$$

This is a central part of the noncommutative class field theory, and the better the description of f_i, the more information one obtains about the Galois extensions. Liftings f_i were defined in [2, Def. 3–4] by using arbitrary topological $\mathbb{Z}_p$-generators of $U_{\mathcal{E}_i}^{\sigma_i-1}$.

DEFINITION. Call an extension L/F *regular* if $\mathcal{L} \setminus \mathcal{F}$ contains no primitive pth root. In positive characteristic every extension is regular. In characteristic zero L/F is regular if and only if the extension $F(\zeta_p)/F$ is unramified or the extension $L(\zeta_p)/L$ is not unramified. In particular, if $E_1 = E_0$ then L/F is regular.

Now we make a correction for the paragraph standing between Definition 3 and Definition 3′ in the published version of [2]. The statement there holds for regular extensions. Indeed, let $\mathcal{F}$ be of characteristic zero. If a primitive pth root of unity ζ_p equals $u^{\sigma_1 - 1}$ with $u \in U_{\mathcal{E}_1}$, then $N_{\mathcal{E}_1/\mathcal{E}_0}(\zeta_p) = 1$. Hence, if $\mathcal{E}_1 \setminus \mathcal{E}_0$ contains no primitive pth root of unity (i.e. L/F is regular), then so does $U_{\mathcal{E}_1}^{\sigma_1 - 1}$. All the assertions of [2] following Def. 3 hold for regular Galois arithmetically profinite extensions.

In the general case of (non-regular) extensions the group $U_{\mathcal{E}_1}^{\sigma_1 - 1}$ may have a nontrivial p-torsion (for example, if $E_0 = \mathbb{Q}_p$ and $E_1 = \mathbb{Q}_p(\zeta_p)$). It is not clear at the moment how to define the corresponding map f_1 for non-regular extensions.

Below we give a new definition of f_i for regular Galois arithmetically profinite extensions L/F, $L \subset F^{\varphi}$.

3. *Splitting exact sequences*

The following theorem leads to a new definition of liftings f_i.

For submodules M_i of $U_{\mathcal{E}_j}$ denote by $\prod M_i$ their product.

Denote $E = E_k$, $E' = E_{k+1}$, $\mathcal{E} = \mathcal{E}_k$, $\mathcal{E}' = \mathcal{E}_{k+1}$. Denote $\sigma = \sigma_{k+1}$.

Recall that in the abelian class field theory an important role is played by the following exact sequence

$$1 \longrightarrow T \longrightarrow U_{\mathcal{E}'}/U_{\mathcal{E}'}^{\sigma-1} \xrightarrow{N_{\mathcal{E}'/\mathcal{E}}} U_{\mathcal{E}} \longrightarrow 1.$$

Here T is the isomorphic image of Gal(E'/E) in $U_{\mathcal{E}'}/U_{\mathcal{E}'}^{\sigma-1}$ with respect to the homomorphism

$$\mathrm{Gal}(E'/E) \longrightarrow U_{\mathcal{E}'}/U_{\mathcal{E}'}^{\sigma-1}, \quad \rho \longmapsto \pi_{E_{k+1}}^{\rho-1} U_{\mathcal{E}'}^{\sigma-1},$$

see [4, Ch.IV (1.7)].

THEOREM. *Fix $k \geqslant 1$. Assume that L/F is a regular extension.*

Denote by T' the intersection of T with $\left(\prod_{i\leqslant k} U_{\mathcal{E}'}^{\sigma_i-1}\right)U_{\mathcal{E}'}^{\sigma-1}/U_{\mathcal{E}'}^{\sigma-1}$. We have an exact sequence

$$1 \longrightarrow T' \longrightarrow \left(\prod_{i\leqslant k} U_{\mathcal{E}'}^{\sigma_i-1}\right)U_{\mathcal{E}'}^{\sigma-1}/U_{\mathcal{E}'}^{\sigma-1} \xrightarrow{N_{\mathcal{E}'/\mathcal{E}}} \prod_{i\leqslant k} U_{\mathcal{E}}^{\sigma_i-1} \longrightarrow 1.$$

The sequence splits by a (not uniquely determined in general) homomorphism

$$f\colon \prod_{i\leqslant k} U_{\mathcal{E}}^{\sigma_i-1} \longrightarrow \left(\prod_{i\leqslant k} U_{\mathcal{E}'}^{\sigma_i-1}\right)U_{\mathcal{E}'}^{\sigma-1}/U_{\mathcal{E}'}^{\sigma-1}.$$

Proof. It is convenient to divide it into several parts.

1. The product of modules $\prod_{i\leqslant k} U_{\mathcal{E}}^{\sigma_i-1}$ is a closed $\mathbb{Z}_p$-submodule of $U_{1,\mathcal{E}}$. Let λ_j be a system of topological multiplicative generators of the topological $\mathbb{Z}_p$-module $\prod_{i\leqslant k} U_{\mathcal{E}}^{\sigma_i-1}$ which satisfy the following property: if the torsion of this group is nontrivial, it includes λ_* of order p^m, and the rest of λ_j are topologically independent over $\mathbb{Z}_p$.

Define a map f on the topological generators λ_j as

$$f(\lambda_j) = u_j\, U_{\mathcal{E}'}^{\sigma-1}$$

where u_j is any element of $\left(\prod_{i\leqslant k+1} U_{\mathcal{E}'}^{\sigma_i-1}\right)$ whose norm equals λ_j. We will prove by the end of the fifth part that $f(\lambda_*)^{p^m} \in U_{\mathcal{E}'}^{\sigma-1}$. Hence we can extend f to a homomorphism $f\colon \prod_{i\leqslant k} U_{\mathcal{E}}^{\sigma_i-1} \longrightarrow \prod_{i\leqslant k+1} U_{\mathcal{E}'}^{\sigma_i-1}/U_{\mathcal{E}'}^{\sigma-1}$ which is a section of the exact sequence in the theorem.

Suppose that $m > 0$, i.e. λ_*, different from 1, is in the system of the generators. As discussed at the end of the previous section, if $k = 1$ and L/F is a regular extension then $m = 0$. Hence $k > 1$.

We claim that then s_{k+1} (defined in section 2) is prime to p. This will be proved by the end of the fourth part.

2. By [4, Ch.III (2.3)] we know that if s_{k+1} is divisible by p then $s_{k+1} = pe(E_k)/(p-1)$, a primitive pth root lies in E_k and there is a prime element π_k of E_k such that $E_{k+1} = E_k(\sqrt[p]{\pi_k})$. Using [4, Ch.II Prop. 4.5] we deduce that s_i are divisible by p for $2 \leqslant i \leqslant k+1$. So all the ramification breaks s_i, $2 \leqslant i \leqslant k+1$ take their maximal possible values. Using local class field theory and looking at the norm group of E_k/E_1 it is easy to see that E_k/E_1 is a cyclic extension (see, e.g. [1, Prop. 1.5]). Then $\sigma_2|_E$ is a generator of its Galois group. Recall that we assume that the torsion element belongs to the system of generators of $\prod U_{\mathcal{E}}^{\sigma_i - 1}$. Hence a primitive pth root ζ_p can be written as $u_1^{\sigma_1-1}u_2^{\sigma_2-1}$ with $u_i \in U_{\mathcal{E}}$. We will show by the end of the fourth part that this leads to a contradiction; then s_{k+1} is prime to p.

3. Denote by v the discrete valuation of $\mathcal{E}$ and let π be a prime element of E. To get a contradiction, choose u_1 with maximal possible value of $v(u_1 - 1)$ such that $\zeta_p = u_1^{\sigma_1-1}u_2^{\sigma_2-1}$. We will show that we can increase the value $v(u_1-1)$, and this gives a contradiction.

Using the description of the norm map in [4, Ch.III sect.1] we deduce that

$$\pi^{\sigma_1} = \theta_1\pi + \text{ terms of higher order},$$
$$\pi^{\sigma_2 - 1} = 1 + \theta_2\pi^{e(E_2)/(p-1)} + \text{ terms of higher order},$$

with non-zero multiplicative representatives θ_i, θ_1 is a primitive lth root.

The Galois group of E/F is the semi-product of cyclic groups of order $l = |E_1 : E_0|$ and $|E_k : E_1|$. Let R be the fixed field of the first group. Then $\sigma_1|_E$ as a generator of the Galois group of E/R. Denote $\mathcal{R} = R\mathcal{E}$.

Let θ run through non-zero multiplicative representatives. In the first choice of representatives in $U_{\mathcal{E}}$ of the quotients U_i/U_{i+1} of the group of principal units of $\mathcal{E}$ we can include in it units $1 + \theta\pi_R^j$ where π_R is a prime element of R. Note that σ_1 acts trivially on such elements. In addition,

$$(1+\theta\pi^i)^{\sigma_1 - 1} = 1 + \theta(\theta_1^i - 1)\pi^i + \text{ terms of higher order}, \qquad \text{if } (i,l) = 1.$$

In the second choice of topological generators of the group of principal units of $\mathcal{E}$ take elements $1+\theta\pi^i$, $(i,p) = 1$, $i < pe(E)/(p-1)$ and an appropriate element $1 + \theta_*\pi^{pe(E)/(p-1)}$ (see, e.g. [4, Ch.I sect.6]). We get

$$(1+\theta\pi^i)^{\sigma_2 - 1} = 1 + i\theta\theta_2\pi^{i+e(E_2)/(p-1)} + \text{ terms of higher order}, \qquad \text{if } (i,p) = 1.$$

4. From the description of the behaviour of the map $x \mapsto x^p$ on the group of principal units (see, e.g., [4, Ch.I sect.5]) we deduce the following. If for some $r \geqslant 0$ the element $\big((1+\theta\pi^i)^{p^r}\big)^{\sigma_2 - 1}$ with $(i,p) = 1$ is not closer to 1 than ζ_p, then

$$\big((1+\theta\pi^i)^{p^r}\big)^{\sigma_2-1} = 1 + (i\theta\theta_2)^{p^r}\pi^{p^r(i+e(E_2)/(p-1))} + \text{ terms of higher order}, \qquad (i,p) = 1.$$

Since $\zeta_p \in \mathcal{E}_1$, l divides $e(E_2)/(p-1)$. From the previous description of the action of σ_1 we deduce that $v(u_1^{\sigma_1-1} - 1) = v(\zeta_p - 1)$ does not hold. Using the description of the action of σ_2 and observing that $p^r(i + e(E_2)/(p-1)) = p^{k-2}e(E_2)/(p-1) = v(\zeta_p - 1)$ for $r \geqslant 0$ implies p divides i, we also deduce that $v(u_2^{\sigma_2-1} - 1) = v(\zeta_p - 1)$ does not hold.

Hence $v(u_1^{\sigma_1-1} - 1) = v(u_2^{\sigma_2-1} - 1) < v(\zeta_p - 1)$ and $u_2^{\sigma_2-1} = 1 + (i\theta\theta_2)^{p^r}\pi^{p^r(i+e(E_2)/(p-1))} +$ terms of higher order, for some $r \geqslant 0$. Denote $j = p^r(i + e(E_2)/(p-1))$. Then $u_1^{\sigma-1}$ must start with

$1-(i\theta\theta_2)^{p^r}\pi^j$, and hence j is not divisible by l. Due to the choice of u_1 we can assume that when it is presented as the product of the first choice of representatives in the group of principal units of $\mathcal{E}$, that product does not contain elements from $\mathcal{R}$. Therefore $u_1 = w^{\sigma_2-1}u_1'$, $v(u_1'-1) > v(u_1-1)$, where $w = (1+\eta\pi^i)^{p^r}$, $\eta^{p^r} \equiv -\theta^{p^r}(\theta_1^j-1)^{-1} \mod \pi$ and $w^{\sigma_2-1} = 1-(i\theta\theta_2)^{p^r}(\theta_1^j-1)^{-1}\pi^j+$ terms of higher order.

Now $\zeta_p = u_1^{\sigma_1-1}u_2^{\sigma_2-1} = {u_1'}^{\sigma_1-1}{u_2'}^{\sigma_2-1}$ where $u_2' = u_2 w^{\sigma_1-1}z$. Here $z=1$ is E_k/E_0 is abelian and $z = (w^{\sigma_1\sigma_2})^{1+\dots+\sigma_2^{r-2}} \in U_{\mathcal{E}}$ where $(\sigma_1^{-1}\sigma_2\sigma_1)|_E = \sigma_2^r|_E$, $r>1$, otherwise. Since $v(u_1'-1) > v(u_1-1)$, we get a contradiction.

Thus, s_{k+1} is prime to p.

5. Now, we argue similarly to the proof of a part of [2, Lemma 3]. Denote $\beta_* = u_*^{p^m}$. We aim to show that $\beta_* \in U_{\mathcal{E}'}^{\sigma-1}$. We get $N_{\mathcal{E}'/\mathcal{E}}\beta_* = 1$, hence β_* can be written as $\pi_{E'}^{\rho-1}u^{\sigma-1}$ with $\rho \in \mathrm{Gal}(E'/E)$, $u \in U_{\mathcal{E}'}$, $\pi_{E'}$ a prime element of E'. We shall show that $\rho = 1$. Then $u_*^{p^m}$ belongs to $U_{\mathcal{E}'}^{\sigma-1}$, as desired.

Find a unit δ in $\mathcal{E}'$ such that $\delta^{1-\varphi} = u_*^{p^{m-1}}$. Then, as briefly discussed in section 1, the reciprocity homomorphism for E'/E maps ρ to $(N_{\mathcal{E}'/\mathcal{E}}\delta)^p \mod N_{E'/E}U_{E'}$; for more detail see [4, Ch.IV sect.3]. If $\varepsilon = N_{\mathcal{E}'/\mathcal{E}}\delta$ belongs to E, then the image of ρ belongs to $N_{E'/E}U_{E'}$, and hence, since the reciprocity homomorphism is injective for abelian extensions, $\rho = 1$. If ε does not belong to E, then, since $\varepsilon \in \mathcal{E}$, we can write $\varepsilon^p = a^p\omega$ where $a \in U_E$ and $\omega \in U_E$ is a p-primary element (i.e. the extension $E(\sqrt[p]{\omega})/E$ is unramified of degree p). Since s_{k+1} is prime to p, we have $s_{k+1} < v(\omega-1) = pe(E)/(p-1)$. Properties of the norm map (see e.g. [4, Ch. III sect. 1]) imply that $\omega \in N_{E'/E}U_{E'}$. Therefore the image of ρ, which is the class of ε^p, belongs to $N_{E'/E}U_{E'}$. Thus, $\rho = 1$, as desired. □

Remark 1. The sequence $1 \to T \to U_{\mathcal{E}'}/U_{\mathcal{E}'}^{\sigma-1} \xrightarrow{N_{\mathcal{E}'/\mathcal{E}}} U_{\mathcal{E}} \to 1$ does not split if and only if s_{k+1} is divisible by p (i.e. the extension E_{k+1}/E_k is not of Artin–Schreier type). This follows from the fifth part of the proof of the previous theorem.

Remark 2. $T' = \{1\}$ if and only if the extension E'/F is abelian; in this case the splitting f is uniquely determined.

Remark 3. The sequence $1 \to T' \to \left(\prod_{i\leqslant k+1} U_{\mathcal{E}'}^{\sigma_i-1}\right)/U_{\mathcal{E}'}^{\sigma-1} \xrightarrow{N_{\mathcal{E}'/\mathcal{E}}} \prod_{i\leqslant k} U_{\mathcal{E}}^{\sigma_i-1} \to 1$ splits in the category of $\mathbb{Z}_p$-modules, but not necessarily in the category of pro-algebraic modules (see also Remark 5).

4. *A new definition of f_i and $Y_{L/F}$*

We assume in this section that the reader has a good knowledge of [2].

Definition. Using the previous theorem, we introduce homomorphisms ($k \geqslant 1$)

$$h_k\colon \prod_{1\leqslant i\leqslant k} U_{\mathcal{E}_k}^{\sigma_i-1} \longrightarrow \Big(\prod_{1\leqslant i\leqslant k+1} U_{\mathcal{E}_{k+1}}^{\sigma_i-1}\Big)/U_{\mathcal{E}_{k+1}}^{\sigma_{k+1}-1}.$$

Set $X_i = U_{\mathcal{E}_i}^{\sigma_i-1}$. Let $g_k\colon \prod_{1\leqslant i\leqslant k} U_{\mathcal{E}_k}^{\sigma_i-1} \longrightarrow \prod_{1\leqslant i\leqslant k+1} U_{\mathcal{E}_{k+1}}^{\sigma_i-1}$ be any map such that $h_k = g_k \mod U_{\mathcal{E}_{k+1}}^{\sigma_{k+1}-1}$.

Define

$$f_i \colon X_i \longrightarrow U_{\mathcal{N}(L/E_i)} \longrightarrow U_{\mathcal{N}(L/F)}$$

as any map such that its $\mathcal{E}_j$-component for $j > i$ coincides with $(g_{j-1} \circ \cdots \circ g_i)|_{X_i}$.

This definition of f_i, since it comes from the splitting *homomorphisms* in the previous theorem, is more functorial than that in [2].

With this choice of f_i [2, Lemma 4] holds for all regular extensions. Denote by Z_i the image of f_i. Set

$$Z_{L/F} = Z_{L/F}(\{E_i, f_i\}) = \left\{ \prod_i z^{(i)} : z^{(i)} \in Z_i \right\}.$$

Define

$$Y_{L/F} = \{y \in U_{\mathcal{N}(L/F)} : y^{1-\varphi} \in Z_{L/F}\}.$$

As in [2], the map $1 - \varphi$ induces an isomorphism between the group $U^{\diamond}_{\mathcal{N}(L/F)}/Y_{L/F}$ and group $\ker N_{\mathcal{L}/\mathcal{F}}/Z_{L/F}$.

The following theorem is proved exactly in the same way as [2, Th. 1 and Th. 2].

Theorem. *Let L/F be a good Galois arithmetically profinite extension.*

The map $\mathrm{Gal}(L/F) \longrightarrow \ker N_{\mathcal{L}/\mathcal{F}}/Z_{L/F}, \quad \tau \longmapsto X^{\tau-1}$ *is a bijection.*
For every $U \in U^{\diamond}_{\mathcal{N}(L/F)}$ there is a unique automorphism $\tau \in \mathrm{Gal}(L/F)$ satisfying

$$U^{1-\varphi} \equiv X^{\tau-1} \bmod Z_{L/F}.$$

Thus the map

$$\mathcal{N}_{L/F} \colon \mathrm{Gal}(L/F) \longrightarrow U^{\diamond}_{\mathcal{N}(L/F)}/Y_{L/F}, \quad \tau \longmapsto U$$

where $U \in U^{\diamond}_{\mathcal{N}(L/F)}/Y_{L/F}$ satisfies the equation of the previous paragraph, is a bijection.

Remark 4. Thus we get the second reciprocity map $\mathcal{H}_{L/F} \colon U^{\diamond}_{\mathcal{N}(L/F)} \longrightarrow \mathrm{Gal}(L/F)$ defined by $\mathcal{H}_{L/F}(U) = \tau$. The above construction of $Z_{L/F}$ and $Y_{L/F}$ provides a new calculation of its kernel.

Remark 5. The group $Z_{L/F}$ for a finite extension L/F is a subgroup of finite index of $V(\mathcal{L}/\mathcal{F})$. Recall [10] that $V(\mathcal{L}/\mathcal{F})$ is the connected component of $\ker N_{\mathcal{L}/\mathcal{F}}$ in the pro-algebraic Zariski topology. The group $Z_{L/F}$ is a connected subgroup of finite index of $V(\mathcal{L}/\mathcal{F})$; and one can show that the quotient $V(\mathcal{L}/\mathcal{F})/Z_{L/F}$ has exponent $\leqslant p$.
It is a challenging problem to investigate if one can modify the Zariski topology to a new topology t so that $Z_{L/F}$ becomes the connected component of $\ker N_{\mathcal{L}/\mathcal{F}}$; then one would have a bijection between $\pi_1^t(U_{\mathcal{F}})/N_{L/F}\pi_1^t(U_{\mathcal{L}})$ and $\mathrm{Gal}(L/F)$ similarly to the (geometric) abelian case.

Remark 6. It is an open problem if the subgroup $Z_{L/F}$ depends on the choice of the tower E_i for nonabelian extensions L/F.

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