# On the IUT theory of Shinichi Mochizuki

Ivan Fesenko

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also known as arithmetic deformation theory.

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IUT is not only a new area in mathematics. It already includes the proof of one its main applications:

For an integer $n = \pm \prod p_i^{m_i}$ denote $\mathrm{red}(n) = \prod p_i$ (the reduced part).

A version of abc conjecture

*there is a positive integer m such that for every $\varepsilon > 0$ there is a positive $\kappa \in \mathbb{R}$ such that for every three non-zero coprime integers $a, b, c$ satisfying $a + b = c$, the inequality*

$$\boxed{\max(|a|, |b|, |c|) < \kappa\, \mathrm{red}(abc)^{m+\varepsilon}}$$

*holds.*

In some of the strongest versions of the abc conjectures $m$ is 1.
In some naive approaches $\kappa$ was expected to be close to 1.

Abc conjectures describe a kind of fundamental balance between addition and multiplication, formalising the observation that
*when two positive integers a and b are divisible by large powers of small primes, $a + b$ tends to be divisible by small powers of large primes.*

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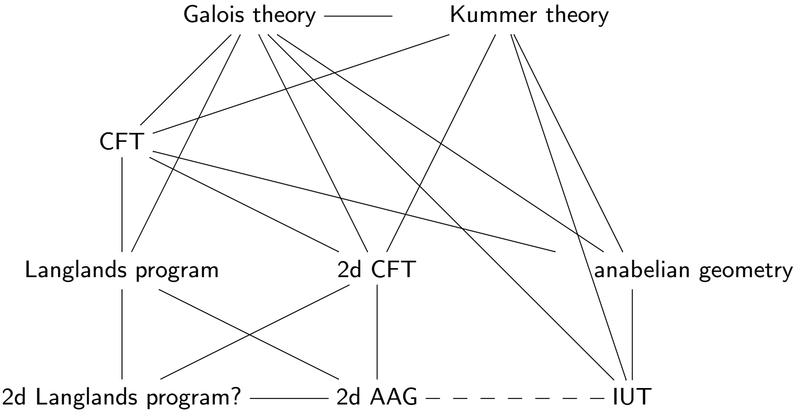
# The place of IUT

CFT = class field theory
AAG = adelic analysis and geometry
2d = two-dimensional (i.e. for arithmetic surfaces)



Shinichi Mochizuki

Inter-universal Teichmüller theory (IUT)

Also known as Arithmetic Deformation Theory (ADT)

Preprints 2012–2017

I: Constructions of Hodge theaters

II: Hodge–Arakelov-theoretic evaluation

III: Canonical splittings of the log-theta-lattice

IV: Log-volume computations and set-theoretic foundations

# Introductory texts and surveys of IUT

The order is chronological,
The numbers may or may not correspond to easiness of reading (5 is very easy)

2. A Panoramic Overview of Inter-universal Teichmüller Theory, by Shinichi Mochizuki

4. Arithmetic Deformation Theory via Algebraic Fundamental Groups and Nonarchimedean Theta-Functions, Notes on the Work of Shinichi Mochizuki, by Ivan Fesenko

1. Introduction to Inter-universal Teichmüller Theory (in Japanese), by Yuichiro Hoshi

3. The Mathematics of Mutually Alien Copies: From Gaussian Integrals to Inter-universal Teichmüller Theory, by Shinichi Mochizuki

5. Fukugen, by Ivan Fesenko

Materials of two workshops on IUT

Oxford Workshop on IUT Theory of Shinichi Mochizuki, December 7-11 2015
RIMS workshop on IUT Summit, July 18-27 2016

can be useful in its study.

Total number of participants of the two workshops: more than 100.
They included geometers and logicians.

The workshops helped to increase the number of people actively studying IUT from 4 in 2014 to 15 in 2017.

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# On IUT very briefly (hence, vaguely)

Algebraic geometry involves locally the correspondence between affine varieties and commutative rings. The most common picture in Grothendieck's volumes is a commutative diagram of commutative rings and homomorphisms and a similar one for local and global geometric objects.

Anabelian geometry for hyperbolic curves over number fields and other fields, as proved by Mochizuki 25-15 years ago, is a correspondence between these geometric objects and their arithmetic fundamental groups (or slightly more complicated objects).

Fundamental groups are highly non-commutative, but they have one algebraic operation, not two. This opens the perspective to try to perform deformations of these geometric objects not seen by algebraic geometry, using the fact that there are more maps, group homomorphisms and variations of those between topological groups in comparison to morphisms between commutative rings.

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However, when one starts to play with basic diagrams of topological groups and maps between them, one immediately sees that the diagrammes are rarely commutative.

This obstruction was already seen 20 years ago by several researchers in anabelian geometry. The main new contribution of Mochizuki in the IUT theory for certain hyperbolic curves (e.g. an elliptic curve minus 1 point) is a new fundamental understanding of how to bound from above the lack of commutativity of certain crucial diagrammes of arithmetic fundamental groups and certain maps between them using anabelian geometry and various symmetries associated to such curves. This bound is then translated into the bound in abc type inequalities.

IUT is a certain categorical monoidal geometry with few commutative diagrams but tools to measure their deviations from commutativity in certain situations which results in the new arithmetic deformation theory that is entirely unavailable via the standard arithmetic geometry.

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# On IUT very briefly

IUT works with deformations of multiplication.

These deformations are not compatible with ring structure.

Deformations are coded in theta-links between theatres which are certain systems of categories associated to an elliptic curve over a number field.

Ring structures do not pass through theta-links.

Galois and fundamental groups (groups of symmetries of rings) do pass.

To restore rings from such groups (which pass through a theta-link) one uses anabelian geometry results about number fields and hyperbolic curves over them.

Measuring the result of the deformation produces bounds which eventually lead to solutions of several famous problems in number theory.

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It is crucial that such highly non-linear objects as full absolute Galois and algebraic fundamental groups are used in IUT.

Unlike in two other generalisations of class field theory, the use of more linear objects such as the maximal abelian quotient or quotients related to the study of representations of these groups is *not sufficient for arithmetic deformation theory*.

# List of main concepts of IUT

- ◦ mono-anabelian geometry, mono-anabelian reconstruction
- ◦ categorical geometry including frobenioids
- ◦ noncritical Belyi maps and their applications
- ◦ Belyi cuspidalisation and elliptic cuspidalisation and their applications
- ◦ mono-theta-environment
- ◦ generalised Kummer theory and multi-radial Kummer detachment
- ◦ principle of Galois evaluation
- ◦ rigidities (discrete rigidity, constant multiple rigidity, cyclotomic rigidity)
- ◦ mono-anabelian transport
- ◦ coric functions to transport elements of number fields
- ◦ multiradiality and indeterminacies
- ◦ (Hodge) theatres
- ◦ theta-link and two types of symmetry
- ◦ log-link, log-shell
- ◦ log-theta-lattice

# Other applications of IUT

IUT proves versions of two other conjectures, by Szpiro and Vojta.
This implies a certain version of the abc inequality which is effective for odd primes.

**The strong Szpiro conjecture over number fields**

*For every $\varepsilon > 0$
there is a positive real number $\kappa$, depending on $\varepsilon$,
such that
for all number fields $K$ and all elliptic curves $E$ over $K$ the inequality*

$$\boxed{D_E < \kappa (C_E D_K)^{6+\varepsilon}}$$

*holds, where
$D_E$ is the norm of the minimal discriminant of $E$,
$C_E$ is the norm of the conductor of $E$,
$D_K$ is the absolute value of the absolute discriminant of $K$.*

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# The Szpiro inequality in the geometric case

Over $\mathbb{C}$, the property analogous to the Szpiro conjecture deals with

a smooth projective surface

equipped with a structure of non-split minimal elliptic surface

fibred over a smooth projective connected complex curve of genus $g$,

such that the fibration admits a global section,

and each singular fibre of the fibration is $n_i$ projective lines

which intersect transversally with two neighbouring lines (an $n_i$-gon).

The geometric Szpiro inequality:

$$S \leq 6(2g - 2 + N)$$

*where $S =$ the sum of the number of components of singular fibres,*
*$N =$ the number of singular fibres.*

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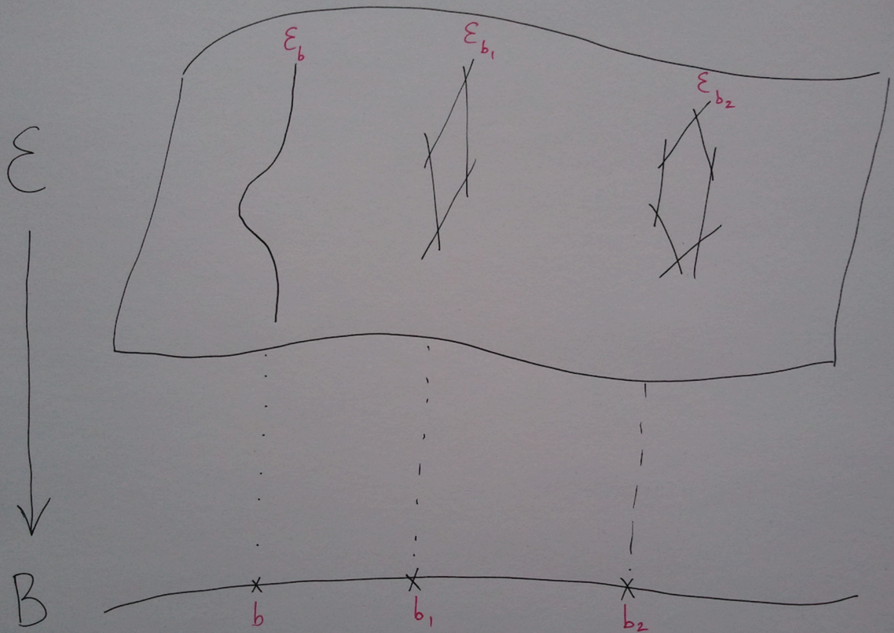

$\mathcal{E}_b$
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There are many proofs of the geometric Szpiro inequality. It was already known to Kodaira.

The nearest to IUT is a proof by Bogomolov (extended by Zhang) which uses the hyperbolic geometry of the upper half-plane, $SL_2(\mathbb{Z})$ and $SL_2(\mathbb{R})$.

As noticed by Kremnitzer, essential part of Bogomolov's proof was already known to Milnor, and, after the work of Gromov, can be interpreted as a result on bounded cohomology and the bounded Euler class.

The Bogomolov proof introduces a certain rotation of the $n_i$-gons and proves that these rotations are synchronised, like windmills revolving in synchrony in the presence of wind.

See S. Mochizuki, Bogomolov's proof of the geometric version of the Szpiro conjecture from the point of view of inter-universal Teichmüller theory, Res. Math. Sci. 3(2016), 3:6

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# On anabelian geometry: fields

Let $K^{\mathrm{alg}}$ be an algebraic closure of a number field $K$.

The group of symmetries, the absolute Galois group of $K$ is

$$G_K = \mathrm{Aut}_K K^{\mathrm{alg}},$$

the group of ring automorphisms of $K^{\mathrm{alg}}$ over $K$.

Neukirch–Ikeda–Uchida theorem:

*For two number fields $K_1, K_2$*

*every isomorphism of topological groups $\lambda\colon G_{K_1} \xrightarrow{\sim} G_{K_2}$*

*comes from a unique field isomorphism $\sigma\colon K_2^{\mathrm{alg}} \xrightarrow{\sim} K_1^{\mathrm{alg}}$, $\sigma(K_2) = K_1$:*

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This theorem does not work if global fields are replaced by local fields: finite extensions of $\mathbb{Q}_p$.

However,

Theorem: *Let $F_1, F_2$ be two finite extensions of $\mathbb{Q}_p$ and let there be a homeomorphism between their absolute Galois groups which is compatible with their upper ramification filtrations. Then the fields are isomorphic.*

This theorem has two different proofs, by Mochizuki and by Abrashkin.

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# Algebraic fundamental groups

For any geometrically integral scheme $X$ over a perfect field $K$ one has its *algebraic fundamental group* $\pi_1(X)$ defined using étale covers of $X$.

If $C$ is a complex irreducible smooth projective curve minus a finite collection of its points, then $\pi_1(C)$ is isomorphic to the profinite completion of the topological fundamental group of the Riemann surface associated to $C$.

The map $X \to \mathrm{Spec}(K)$ induces the surjective homomorphism

$$\pi_1(X) \to \pi_1(\mathrm{Spec}(K)) = G_K,$$

its kernel is the geometric fundamental group $\pi_1^{\mathrm{geom}}(X)$.

Suppressed dependence of the fundamental groups on basepoints actually means that various objects are well-defined only up to conjugation by elements of $\pi_1(X)$.

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# Anabelian geometry for curves

Let $C$ be a hyperbolic curve over a number field $K$. Denote $L = K(C)$, a 2d global field.

$\pi_1(C) =$ the Galois group of $L_C$ over $L$,

where $L_C$ is the compositum of finite Galois extensions of $L$

such that the corresponding morphism of proper curves is étale over $C$

and each finite Galois subextension of $L_C/L$ comes from a curve étale over $C$.

Conjecture of Grothendieck: *The curve $C$ may be "reconstituted" from the structure of the fundamental group $\pi_1(C)$ as a topological group equipped with its associated surjection to $G_K$.*

Theorem (Mochizuki, 1995): *proof of Grothendieck's conjecture.*

Compare with Rigidity theorem (Mostow–Prasad–Gromov): *the isometry class of a finite-volume hyperbolic manifold of dimension $\geq 3$ is determined by its topological fundamental group.*

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$\pi_1(C) =$ the Galois group of $L_C$ over $L$,

where $L_C$ is the compositum of finite Galois extensions of $L$

such that the corresponding morphism of proper curves is étale over $C$

and each finite Galois subextension of $L_C/L$ comes from a curve étale over $C$.

Conjecture of Grothendieck: *The curve $C$ may be "reconstituted" from the structure of the fundamental group $\pi_1(C)$ as a topological group equipped with its associated surjection to $G_K$.*

Theorem (Mochizuki, 1995): *proof of Grothendieck's conjecture.*

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One of important results in mono-anabelian geometry is

Theorem (Mochizuki, 2008): *one can algorithmically reconstruct a number field $K$ from $\pi_1(C)$ for certain hyperbolic curves or orbi-curves $C$ over $K$, e.g. $E \setminus \{0\}$ or its quotient by $\langle \pm 1 \rangle$.*

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The general case is reduced to the case when

$E$ is an *elliptic curve over a number field* $K$ so that each of its reductions is good or *split multiplicative*,

the 6-torsion points of $E$ are rational over $K$,

$K$ contains a 4th primitive root of unity,

the extension of $K$ generated by the $\ell$-torsion points of $E$ has Galois group over $K$ isomorphic to a subgroup of $GL_2(\mathbb{Z}/\ell\mathbb{Z})$ which contains $SL_2(\mathbb{Z}/\ell\mathbb{Z})$.

IUT works with *hyperbolic curves*: the hyperbolic curve

$$X = E \setminus \{0\}$$

over $K$ and the hyperbolic orbicurve

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# The bounds from IUT

Assume that $v$ is a prime of bad reduction of $E$. Tate's theory shows that

$$E(K_v) = K_v^\times / \langle q_v \rangle.$$

This element $q_v$ (the $q$-parameter of $E$ at $v$) is the analogue of $p_i^{n_i}$ in the abc inequality.

*The goal of IUT is to bound*

$$\deg(q_E) = |K : \mathbb{Q}|^{-1} \sum \log |O_v : q_v O_v|,$$

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# The bounds from IUT

In fact, studying the arithmetic deformation one obtains first an inequality

$$-\deg(q_E) \leq -\deg(\Theta_E)$$

where $\Theta_E$ is a certain theta-data after applying the theta-link, subject to certain indeterminacies, assuming the RHS is not equal to $+\infty$.

Then one obtains an inequality

$$-\deg(\Theta_E) \leq a(\ell) - b(\ell)\deg(q_E)$$

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In more precise terms,

$$\frac{1}{6}\deg(q_E) \leq \left(1 + \frac{2^4 5d}{\ell}\right)\left(\deg(\mathrm{cond}_E) + \deg(\delta_{K/\mathbb{Q}})\right) + 2^{14}3^3 5^2 d\ell + c_\circ,$$

where $c_\circ > 0$ comes from the prime number theorem (over $\mathbb{Q}$), prime number theorem is the only result from analytic number theory used in IUT,
$\delta_{K/\mathbb{Q}}$ is the (absolute) different of $K$,
$d$ is the degree of the field of definition of $E$.

Note the dependence on $\ell$.

To derive the required bound on $\deg(q_E)$, one chooses the prime $\ell$ in the interval $\left(\sqrt{\deg(q_E)}, 5c_*\sqrt{\deg(q_E)}\log(c_* \deg(q_E))\right)$, where $c_* = 2^{13}3^3 5d$.

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# Nonarchimedean theta-function

*Étale theta function theory* uses a *non-archimedean theta-function*

$$\theta(u) = \sum_{n\in\mathbb{Z}} (-1)^n q_v^{n(n-1)/2} u^n = (1-u) \prod_{n\geq 1} \big((1-q_v^n)(1-q_v^n u)(1-q_v^n u^{-1})\big).$$

The obvious functional equation $\theta(u) = -u\theta(q_v u)$ implies

$$q_v^{(m^2-m)/2} = \theta(-1)/\theta(-q_v^m), \qquad m \in \mathbb{Z}.$$

This relation is used to represent powers of $q_v$ as special values of a modified theta-function.

IUT demonstrates that these special values are very special objects.

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# Example of arithmetic deformation

Monoid $M$
$=$ the product
of invertible elements $\overline{O_v}^{\times}$ of the ring of integers of an algebraic closure of $K_v$
and the group generated by non-negative powers of $q_v$, with $G_{K_v}$-action.

Local deformation: fix a positive integer $m$

$$M \to M, \quad \text{units go identically to units,} \quad q_v \mapsto q_v^m.$$

In étale theta function theory one uses truncated Kummer theory of line bundles associated to non-archimedean theta-functions as a bridge between monoid-theoretic structures and tempered fundamental group structures associated to the theta-function.

The theory gives an anabelian construction of theta Kummer classes satisfying *three rigidities*:

*discrete rigidity*: one can deal with $\mathbb{Z}$-translates (as a group of covering transformations on the tempered coverings), as opposed to $\hat{\mathbb{Z}}$-translates, of the theta function and $q$-parameters;

*constant multiple rigidity*: the monoid $\theta_2^{m/\ell}\overline{O_v}^{\times}$ : $m \in \mathbb{N}$, where $\theta_2 = \theta_1(i)/\theta_1(u)$, $\theta_1(u) = -u^{-1}\theta(u^2)$, has, up to $2\ell$-th roots of unity, a *canonical splitting* via a Galois evaluation;

*cyclotomic rigidity*: a rigidity isomorphism between the *exterior cyclotome* arising from the roots of unity of the base field and the *interior cyclotome* arising as a subquotient of the geometric tempered fundamental group.

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# Two types of symmetry associated to a prime $l$

There are *two types of symmetry* for the hyperbolic curves associated to the elliptic curve $E$ over a number field $K$, the choice of prime $\ell$ and the theta structure at a bad reduction prime $v$.

They correspond to the LHS and RHS of the illustration and animation to follow.

Recall

$$X = E \setminus \{0\}, \qquad C = X/\langle \pm 1 \rangle.$$

Let $Y$ be a $\mathbb{Z}$-(tempered) cover of $X$ at $v$, which corresponds to the universal graph-cover of the dual graph of the special fibre.

Let $\underline{X} \longrightarrow X$ be its subcover of degree $l$.

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The symmetries are denoted

$$\mathbb{F}_\ell^{\rtimes\pm} = \mathbb{F}_\ell \rtimes \{\pm 1\}, \quad \mathbb{F}_\ell^{\circledast} = \mathbb{F}_\ell^{\times}/\{\pm 1\},$$

with $\mathbb{F}_\ell$ arising from the $\ell$-torsion points of $E$.

*The first type of symmetry* arises from the action of the geometric fundamental group $(\mathrm{Aut}_K(\underline{X}) \stackrel{\sim}{\to} \mathbb{F}_\ell^{\rtimes\pm})$ and is closely related to the Kummer theory surrounding the theta-values. This symmetry is of an essentially geometric nature and it is additive.

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These symmetries are coded in appropriate *theatres*.
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# Theatres

IUT operates with *(Hodge) theatres* – objects of categorical geometry which generalise some aspects of 1d and 2d adelic objects, taking into account the full fundamental groups and the two symmetries.

Theatres are a system of categories obtained by gluing categories over a base.

Many of the base categories are isomorphic to the full subcategory of finite étale covers of hyperbolic curves.

Very approximately, theatres generalise $\mathbb{A}_K^\times = \prod K_v^\times > K^\times$ with the ideles and global elements inside them
to $T = \prod' T_v > T_0$ with local theatres $T_v$ depending on $K_v$, $E$ and prime $\ell$ and global $T_0$ depending on $K$, $E$ and $\ell$.

Each theatre consists of two portions corresponding to the two symmetries. They are glued together. This gluing is only possible at the level of categorical geometry.

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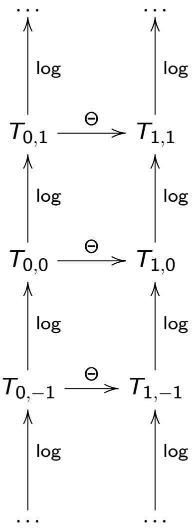
*log-links* and *theta-links* form a noncommutative diagram



There is no natural action of the theta-values on the multiplicative monoid of units modulo torsion, but there is a natural action of the theta-values on the logarithmic image of this multiplicative monoid.

The multiplicative structures on either side of the theta-link are related by means of the value group portions.

The additive structures on either side of the theta-link are related by means of the units group portions, shifted once via the log-link, in order to transform the multiplicative structure of these units group portions into the additive structure.

# Log-theta lattice

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To define the power series logarithm for the log-link one needs to use ring structures, however, the theta-link is not compatible with ring structures.

From the point of view of the codomain of

$$\Theta\colon T_{n,m} \to T_{n+1,m},$$

one can only see the units group and value group portions of the data that appears in the domain of this theta-link.

So one applies the log-link

$$\log\colon T_{n,m-1} \to T_{n,m}$$

in order to give a presentation of the value group portion at $T_{n,m}$ by means of an action of it that arises from applying the log-link to the units group portion at $T_{n,m-1}$.

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in order to give a presentation of the value group portion at $T_{n,m}$ by means of an action of it that arises from applying the log-link to the units group portion at $T_{n,m-1}$.

# Log-theta lattice

One of fundamental ideas of IUT is to consider structures that are invariant with respect to arbitrary vertical shifts $\log\colon T_{n,m-1} \to T_{n,m}$, so called *log-shells*.

A log-shell is a common structure for the log-links in one vertical line.

A key role is played by *Galois evaluation* for special types of functions on hyperbolic curves associated to $E$,
by restricting them to the decomposition subgroups of closed points of the curves via sections of arithmetic fundamental groups,
to get special values, such as theta-values and elements of number fields, acting on log-shells.

Log-shells are shown as central balls, and Galois evaluation is shown in two panels of the illustration and animation to follow.

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# Multiradiality

IUT applies mono-anabelian reconstruction algorithms to algebraic fundamental groups that appear in one universe in order to obtain descriptions of objects constructed from such algebraic fundamental groups that make sense in another universe.

The IUT papers use the terminology of a wheel, its core and spokes.

A functorial algorithm from a radial category to the core category is called *multiradial* if it is full.

Thus, multiradial algorithm expresses objects constructed from a given spoke in terms of objects that make sense from the point of view of other spokes: there is a parallel transport isomorphism between two collections of radial data that lifts a given isomorphism between collections of underlying coric data.

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# Indeterminacies

To obtain *multiradial algorithms*, it may be necessary to allow some *indeterminacies* in the descriptions that appear in the algorithms of the objects constructed from the given spoke.

There are *three indeterminacies*,
shown as Ind 1 – Ind 3 at the illustration and video.

*The first indeterminacy* is closely related to the action of automorphisms of the absolute Galois group of a local field, it corresponds to compatibility with the permutation symmetries of the Galois and arithmetic fundamental groups associated with vertical lines of the log-theta-lattice.

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# Indeterminacies

*The second indeterminacy* is related to the action of a certain compact group of isometries on the logarithmic image of units. It comes from the requirement of compatibility with the horizontal theta-link.

*The third indeterminacy* comes from a certain compatibility of the Kummer isomorphism with the log-links associated to a single vertical line of the log-theta-lattice.

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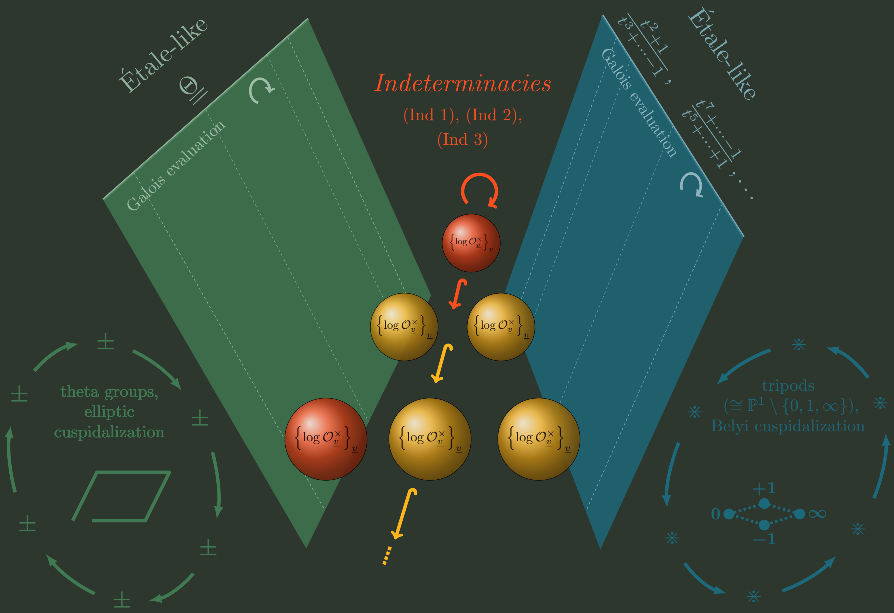
Indeterminacies
(Ind 1), (Ind 2),
(Ind 3)
Étale-like
Θ
Galois evaluation
Étale-like
Galois evaluation
$\frac{t^2+1}{t^3+\cdots-1}$,
$\frac{t^7+\cdots-1}{t^5+\cdots+1}$, . . .
$\{\log \mathcal{O}_{\underline{v}}^{\times}\}_{\underline{v}}$
theta groups,
elliptic
cuspidalization
tripods
($\cong \mathbb{P}^1 \setminus \{0, 1, \infty\}$),
Belyi cuspidalization
+1
0
∞
−1

# Animations related to IUT

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## Various guides on IUT

See this page and this page of Shinichi Mochizuki