Class Field Theory – Its Centenary and Prospect
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Noncommutative local reciprocity maps

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There are several approaches to the reciprocity map, the essence of class field theory, which links the maximal abelian quotient (or sometimes the maximal abelian pro-p-quotient) of the absolute Galois group of a particular field with an appropriate abelian object associated to the field such that certain functorial properties hold.

One of those approaches originates from works of Dwork [D], Serre [S], Hazewinkel [H1], [H2], Iwasawa [I1], [I2] and Neukirch [N1], [N2]. Recall it briefly.

Let F be a local field with finite residue field. Let $F^{\rm ur}$ be the maximal unramified extension of F and let $\widehat{F}$ be the completion of $F^{\rm ur}$. For a separable extension L of F put $L^{\rm ur} = LF^{\rm ur}$, $\widehat{L} = L\widehat{F}$.

For an element σ of $\mathrm{Gal}(L/F)$ let $\tilde\sigma$ be any element of $\mathrm{Gal}(L^{\rm ur}/F^{\rm ur})$ such that $\tilde\sigma|_L = \sigma$ and $\tilde\sigma|_{F^{\rm ur}}$ is a positive integer power of the Frobenius automorphism $\varphi \in \mathrm{Gal}(F^{\rm ur}/F)$. Let Σ be the fixed field of $\tilde\sigma$; it is a finite extension of F.

Let $\mathrm{Gal}(L/F)^{\rm ab}$ be the maximal abelian quotient of $\mathrm{Gal}(L/F)$.

Define the map [N1], [N2]

$$\boldsymbol{N}\colon \mathrm{Gal}(L/F) \to F^*/N_{L/F}L^*$$

by $\sigma \to N_{\Sigma/F}\pi_\Sigma \bmod N_{L/F}L^*$ where π_Σ is any prime element of Σ. During the conference on class field theory in Tokyo, June 1998, Professor T. Tamagawa informed the author that similar constructions were independently developed by K. Iwasawa. We call $\boldsymbol{N}$ the Neukirch–Iwasawa map.

On the other hand, for a finite Galois totally ramified extension L/F of local fields there is a fundamental exact sequence [Se, (2.3)], [H1, (2.7)]

$$1 \longrightarrow \mathrm{Gal}(L/F)^{\rm ab} \xrightarrow{\ c\ } U_{\widehat{L}}/V(L/F) \xrightarrow{N_{\widehat{L}/\widehat{F}}} U_{\widehat{F}} \longrightarrow 1$$

where $V(L/F)$ is the subgroup of $U_{\widehat{L}}$ generated by elements $u^{\sigma-1}$ with $u \in U_{\widehat{L}}, \sigma \in \mathrm{Gal}(L/F)$ and $c(\sigma) = \sigma(\pi)/\pi \bmod V(L/F)$ for a prime element π of L. Note that the same sequence for the maximal unramified extensions instead of their completion is exact.

Define the Hazewinkel homomorphism [H1], [H2], [I1]

$$\boldsymbol{H}\colon U_F/N_{L/F}U_L \to \mathrm{Gal}(L/F)^{\rm ab}$$

by $\boldsymbol{H}(u) = \sigma$ where $u = N_{\widehat{L}/\widehat{F}}(v)$ and $c(\sigma) = \sigma(\pi)/\pi = v/\varphi(v) \bmod V(L/F)$. This map can be extended to finite and infinite Galois extensions [H1], [H2], [I1].

The shortest way to deduce properties of $\boldsymbol{N}$ and $\boldsymbol{H}$ is to work with both maps simultaneously. For a finite Galois extension L/F the composition $\boldsymbol{H} \circ \boldsymbol{N}$ coincides with the epimorphism $\mathrm{Gal}(L/F) \to \mathrm{Gal}(L/F)^{\mathrm{ab}}$ and the composition $\boldsymbol{N} \circ \boldsymbol{H}$ is the identity map of $U_F/N_{L/F}U_L$. Hence $\boldsymbol{N}$ is an epimomorphism with the kernel equal to the commutator group of the Galois group.

This approach with appropriate modifications and generalizations works well in (p-) class field theories of local fields with perfect residue field [F3, H1], higher local fields [F1], [F2], [F4], complete discrete valuation fields with residue field of characteristic p [F5].

In this paper we shall define noncommutative reciprocity maps for arithmetically profinite Galois extensions of local fields extending the approach discussed above. For Fontaine–Wintenberger's theory of arithmetically profinite extension and fields of norms see [W], [FV,Ch.III,sect.5]. For simplicity we treat the case of totally ramified extensions, however, the constructions of this work can undoubtedly be defined for arbitrary Galois arithmetically profinite extensions, in particular, arbitrary finite Galois extensions of local fields.

We use terminology "the field of norms" for finite extensions as well, meaning just the set of norm-compatible sequences in subextensions. In this case by $U_{N(L/K)}$ we mean the group of norm-compatible sequences in the group of units of subextensions in L/K.

We shall work with maps $\boldsymbol{N}_{L/F}$, $\mathcal{N}_{L/F}$ and $\mathcal{H}_{L/F}$. The map $\boldsymbol{N}_{L/F}$ is a generalization of the map $\mathbf{N}$. It injects the Galois group $\mathrm{Gal}(L/F)$ of a finite or infinite arithmetically profinite totally ramified extension L of a local field F into a certain subquotient $U^{\diamond}_{\widehat{N(L/F)}}/U_{N(L/F)}$ of the group of units $U_{\widehat{N(L/F)}}$ of the field of norms $\widehat{N(L/F)} = N(\widehat{L}/\widehat{F})$ of the arithmetically profinite extension $\widehat{L}/\widehat{F}$ which is a natural $\mathrm{Gal}(L/F)$-module.

The map $\boldsymbol{N}_{L/F}$ is a 1-cocycle. It is compatible with the ramification filtration on the Galois group and the natural filtration on local fields.

We shall study the image of $\boldsymbol{N}_{L/F}$ and show that there is a bijection

$$\mathcal{N}_{L/F}\colon \mathrm{Gal}(L/F) \to U^{\diamond}_{\widehat{N(L/F)}}/Y_{L/F}$$

for a certain subgroup $Y_{L/F}$ of $U^{\diamond}_{\widehat{N(L/F)}}$ which contains $U_{N(L/F)}$. To check the properties of $\mathcal{N}_{L/F}$ we shall define a map

$$\mathcal{H}_{L/F}\colon U^{\diamond}_{\widehat{N(L/F)}}/Y_{L/F} \to \mathrm{Gal}(L/F)$$

which acts in the reverse direction. The latter is a generalization of the fundamental exact sequence.

The set $U^{\diamond}_{\widehat{N(L/F)}}/Y_{L/F}$ with a new group structure given by

$$x \star y = x\mathcal{N}^{-1}_{L/F}(x)(y)$$

is isomorphic to $\mathrm{Gal}(L/F)$.

Recall that the field of norms $\widehat{N(L/F)}$ is isomorphic to $\mathbb{F}_p^{\mathrm{sep}}((X))$, so $U_{\widehat{N(L/F)}}$ is isomorphic to $\mathbb{F}_p^{\mathrm{sep}}[[X]]^*$. Thus, every Galois group of a totally ramified arithmetically profinite extension L/F in isomorphic to a certain subquotient of $\mathbb{F}_p^{\mathrm{sep}}[[X]]^*$ which is endowed with the new (noncommutative in general) group structure on it.

The classical abelian reciprocity isomorphism is the $\widehat{F}$-component of the $\mathcal{N}_{L/F}$ and $\mathcal{H}_{L/F}$. If R/F is the maximal abelian subextension of L/F, then the $\widehat{R}$-component of the $\mathcal{N}_{L/F}$ and $\mathcal{H}_{L/F}$ is in fact the metabelian reciprocity map introduced by Koch and de Shalit [K], [KdSh].

Let F be a local field with finite residue field. Let φ in the absolute Galois group G_F of F be an extension of the Frobenius automorphism of the maximal unramified extension F^{ur} over F.

Let F^{φ} be the fixed field of φ. It is a totally ramified extension of F and its compositum with F^{ur} coincides with the maximal separable extension of F. We shall work with Galois extensions of F inside F^{φ}. The reason why in the noncommutative theory one is deemed to work with extensions inside F^{φ} is explained in [KdSh, 0.2].

From abelian local class field theory and a compactness argument one deduces that there is a unique norm-compatible sequence of prime elements (π_E) in finite subextensions of F^{φ}/F, see for instance [KdSh, Lemma 0.2].

Recall that a separable extension L of a local field K is called arithmetically profinite if the subgroup $G_L G_K^x$ is of finite index in G_K for every x (where G_K^x is the upper ramification group of G_K). Equivalently, L/K is arithmetically profinite if it has finite residue field extension and the Hasse–Herbrand function $h_{L/K}(x) = \lim h_{E/K}(x)$ takes real values for all real $x \geqslant 0$ where E/K runs through all finite subextensions in L/K, see [W], [FV, Ch. III, sect. 5]. For an infinite arithmetically profinite extension L/K the field of norms $N = N(L/K)$ is the set of all norm-compatible sequences

$$\{(a_E) : a_E \in E^*, E/K \quad \text{is a finite subextension of } L/K\}$$

and zero, such that the multiplication is componentwise and the addition $(a_E) + (b_E) = (c_E)$ is defined as $c_E = \lim_M N_{M/E}(a_M + b_M)$ where M runs through all finite subextension of E in L. An element of the field of norms has E-component for every finite subextension E/K of L/K. The field N is a local field of characteristic p with the residue field isomorphic to the residue field of L and a prime element $t = (\pi_E)$ which is a sequence of norm-compatible prime elements of finite subextensions of L/K. If the extension L/F is totally ramified, then the discrete valuation $v_{N(L/F)}$ is given by $v_{N(L/F)}((a_E)) = v_F(a_F) = v_E(a_E)$. Every automorphism τ of L over K induces an automorphism τ of the field of norms: $\tau((\pi_E)) = (\tau\pi_E)$. If M is a separable extension of L, then one defines $N(M, L/K)$ as the compositum of all $N(F'/K)$ where F' runs through finite extensions of L in M.

For Fontaine–Wintenberger's theory of fields of norms see [W], [9, Ch. III, sect. 5]. For a finite subextension M/K of an arithmetically profinite extension L/K the extension L/M is arithmetically profinite; for every subextension M/K of an arithmetically profinite extension L/K the extension M/K is arithmetically profinite.

One of the central theorems of the theory of fields of norms tells that the absolute Galois group of $N(L/K)$ coincides with $G(N(L^{\mathrm{sep}}, L/K)/N(L/K))$ and the latter is isomorphic to $G(L^{\mathrm{sep}}/L)$, see [W, 3.2.2]. Every abelian totally ramified extension is arithmetically profinite.

If L/K is finite, then denote by $N(L/K)$ the set consisting of norm-compatible sequences in the multiplicative groups of finite subextensions in L/K and of 0. By $U_{N(L/K)}$ we mean the group of norm-compatible sequences in the group of units of subextensions in L/K.

Let $L \subset F^{\varphi}$ be a Galois (possibly infinite) totally ramified arithmetically profinite extension of F. The canonical sequence of norm-compatible prime elements (π_E) in finite subextensions of F^{φ}/F supplies the canonical sequence of norm-compatible prime elements (π_E) in finite subextensions of L/F and therefore the canonical prime element X of the local field $N(L/F)$. Denote by φ the automorphism of $N(L/F)^{\mathrm{ur}}$ and $\widehat{N(L/F)}$ corresponding to φ.

Using solvability of Galois extensions in the local situation fix a tower of subfields $F = E_0 - E_1 - E_2 - \dots$, such that $L = \cup E_i$, E_i/F is a Galois extension, and E_i/E_{i-1} is cyclic of prime degree p for $i > 1$, E_1/E_0 is cyclic of degree relatively prime to p.

Let $N(\widehat{L}/\widehat{E_i})$ be the field of norms of the arithmetically profinite extension $L\widehat{E_i}/\widehat{E_i}$. It can be identified with the completion $\widehat{N(L/E_i)}$ of the maximal unramified extension $N(L/E_i)^{\mathrm{ur}}$ of $N(L/E_i)$.

For a local field K the symbols $U_K, U_{i,K}$ denote, as usual, the group of units of the ring of integers and the higher groups of units.

DEFINITION 1. Denote by $U^{\diamond}_{\widehat{N(L/F)}}$ the subgroup of the group $U_{\widehat{N(L/F)}}$ of those elements whose $\widehat{F}$-component belongs to U_F.

Recall that every element of the group of units of a local field with separably closed residue field is $(\varphi - 1)$-divisible, see for instance [I2, Lemma 3.11].

To motivate the next definition we interpret the map $\boldsymbol{N}$ for a finite Galois totally ramified extension L/F in the following way. Since in this case both π_Σ and π_L are prime elements of L^{ur}, there is $\varepsilon \in U_{L^{\mathrm{ur}}}$ such that $\pi_\Sigma = \pi_L\varepsilon$. We can take $\tilde{\sigma} = \sigma\varphi$. Then $\pi_L^{\sigma-1} = \varepsilon^{1-\sigma\varphi}$. Let $\eta \in U_{\widehat{L}}$ be such that $\eta^{\varphi-1} = \varepsilon$. Since $(\eta^{\sigma\varphi-1}\varepsilon^{-1})^{\varphi-1} = (\eta^{(\sigma-1)\varphi})^{\varphi-1}$, we deduce that $\varepsilon = \eta^{\sigma\varphi-1}\eta^{(1-\sigma)\varphi}\rho$ with $\rho \in U_L$. Thus, for $\xi = \eta^{\sigma\varphi-1}$

$$\boldsymbol{N}(\sigma) \equiv N_{\Sigma/F}\pi_\Sigma \equiv N_{\widehat{L}/\widehat{F}}\xi \bmod N_{L/F}L^*, \qquad \xi^{1-\varphi} = \pi_L^{\sigma-1}.$$

DEFINITION 2. Define the map

$$\boldsymbol{N}_{L/F}\colon \mathrm{Gal}(L/F) \to U^{\diamond}_{\widehat{N(L/F)}}/U_{N(L/F)}$$

by

$$\boldsymbol{N}_{L/F}(\sigma) = (u_{\widehat{E_i}}) \bmod U_{N(L/F)},$$

where $U = (u_{\widehat{E_i}}) \in U_{\widehat{N(L/F)}}$ satisfies the equation

$$\boxed{U^{1-\varphi} = X^{\sigma-1}}$$

Then, clearly, $(u_{\widehat{E_i}})$ belongs to $U^{\diamond}_{\widehat{N(L/F)}}$ and is defined modulo $U_{N(L/F)}$.

Note that $U^{\diamond}_{\widehat{N(L/F)}}/N_{U(L/F)}$ is the direct product of a quotient group of the group of multiplicative representatives of the residue field of L, a cyclic group $\mathbb{Z}/p^a\mathbb{Z}$ and a countable free topological $\mathbb{Z}_p$-module.

REMARK 1. For a finite extension L/F the $\widehat{F}$-component of $\boldsymbol{N}_{L/F}(\sigma)$ is equal to $N_{\widehat{L}/\widehat{F}}\xi \bmod N_{L/F}U_L$ where $\xi^{1-\varphi} = \pi_L^{\sigma-1}$. In other words, the $\widehat{F}$-component of $\boldsymbol{N}_{L/F}$ is the classical Neukirch–Iwasawa map $\boldsymbol{N}$.

LEMMA 1. *Let M/F be a Galois subextension of L/F and E/F be a finite subextension of L/F. Then the following diagrams of maps are commutative:*

$$\begin{array}{ccc} \mathrm{Gal}(L/E) & \xrightarrow{\boldsymbol{N}_{L/E}} & U^{\diamond}_{\widehat{N(L/E)}}/U_{N(L/E)} \\ \downarrow & & \downarrow \\ \mathrm{Gal}(L/F) & \xrightarrow{\boldsymbol{N}_{L/F}} & U^{\diamond}_{\widehat{N(L/F)}}/U_{N(L/F)}, \end{array}$$

$$\begin{array}{ccc} \mathrm{Gal}(L/F) & \xrightarrow{\boldsymbol{N}_{L/F}} & U^{\diamond}_{\widehat{N(L/F)}}/U_{N(L/F)} \\ \downarrow & & \downarrow \\ \mathrm{Gal}(M/F) & \xrightarrow{\boldsymbol{N}_{M/F}} & U^{\diamond}_{\widehat{N(M/F)}}/U_{N(M/F)}. \end{array}$$

LEMMA 2. *$\boldsymbol{N}_{L/F}$ is injective and $\boldsymbol{N}_{L/F}(\sigma\tau) = \boldsymbol{N}_{L/F}(\sigma)\,\sigma\boldsymbol{N}_{L/F}(\tau)$.*

Proof. If $\boldsymbol{N}_{L/F}(\sigma) = (u_{\widehat{E}_i}) \in U_{N(L/F)}$, then $(u_{\widehat{E}_i})^{\varphi-1} = 1$, so σ acts trivially on the prime elements π_E, therefore $\sigma = 1$. □

REMARK 2. The set $\mathrm{im}(\boldsymbol{N}_{L/F})$ isn't closed in general with respect to the multiplication in $U_{\widehat{N(L/F)}}/U_{N(L/F)}$. However, Lemma 2 implies that being endowed with a new group structure given by

$$x \star y = x\boldsymbol{N}^{-1}_{L/F}(x)(y)$$

$\mathrm{im}(\boldsymbol{N}_{L/F})$ is a group isomorphic to $\mathrm{Gal}(L/F)$.

Let $U^{\diamond}_{n,\widehat{N(L/F)}}$ be the filtration induced from the filtration $U_{n,\widehat{N(L/F)}}$ on the field of norms. For an infinite arithmetically profinite extension L/F with the Hasse–Herbrand function $h_{L/F}$ put $\mathrm{Gal}(L/F)_n = \mathrm{Gal}(L/F)^{h^{-1}_{L/F}(n)}$.

PROPOSITION 1. *$\boldsymbol{N}_{L/F}$ maps*

$$\mathrm{Gal}(L/F)_n \setminus \mathrm{Gal}(L/F)_{n+1} \qquad \textit{into} \qquad U^{\diamond}_{n,\widehat{N(L/F)}}U_{N(L/F)} \setminus U^{\diamond}_{n+1,\widehat{N(L/F)}}U_{N(L/F)}.$$

Proof. Let $\tau \in \mathrm{Gal}(L/F)_n$. Then due to the properties of arithmetically profinite extensions [W, 3.3.2 and 3.3.4] there is a finite subextension Q/F of L/F such that $\pi_{E'}^{\tau-1} \in U_{n,E'}$ for every $E' \supset Q$.

Choose a solution $(u_{\widehat{E}})$ of the equation $(u_{\widehat{E}})^{1-\varphi} = (\pi_E)^{\tau-1}$ such that $u_{\widehat{E'}} \in U_{n,\widehat{E'}}$ for $E' \supset Q$. Then $v_{\widehat{E''}}(((u_{\widehat{E}}) - 1)_{\widehat{E''}}) \geqslant n$ for sufficiently large $E'' \supset Q$ [W, 2.3.2.2, 2.3.2.3]. Hence $(u_{\widehat{E}}) \in U_{n,\widehat{N(L/F)}}$.

If $(\pi_E)^{\tau-1} = (u_{\widehat{E}})^{1-\varphi}$ with $(u_{\widehat{E}}) \in U^{\diamond}_{n+1,\widehat{N(L/F)}}U_{N(L/F)}$, then

$(\pi_E)^{\tau-1} \in U^{\diamond}_{n+1,\widehat{N(L/F)}}$, so $\pi_{E'}^{\tau-1} \in U_{n+1,E'}$ for sufficiently large E' [W,3.2].

Thus, by [W,3.3.2 and 3.3.4], $\tau \in \mathrm{Gal}(L/F)_{n+1}$. □

To study the image of $\boldsymbol{N}_{L/F}$ we shall define after some preliminary considerations a map $\mathcal{H}_{L/F}$ which takes values in $\mathrm{Gal}(L/F)$.

Recall that the norm map is surjective for finite extensions of local fields with separably closed residue field, see for instance [Se, 2.2].

DEFINITION 3. Let σ_i be a generator of $\mathrm{Gal}(E_i/E_{i-1})$. Let $v_{\widehat{E}_i}$ be the discrete valuation of $\widehat{E}_i$. Put $s_i = v_{\widehat{E}_i}(\pi_{E_i}^{\sigma_i-1} - 1)$. Denote $X_i = U_{\widehat{E}_i}^{\sigma_i-1}$. Note that $X_i \leqslant U_{s_i+1,\widehat{E}_i}$. The group X_i

is a $\mathbb{Z}_p$-submodule of $U_{1,\widehat{E}_i}$. It is the direct sum of a cyclic torsion group of order p^{n_i}, $n_i \geqslant 0$, generated by, say, α_i ($\alpha_i = 1$ if $n_i = 0$) and a free topological $\mathbb{Z}_p$-module Y_i. If $\widehat{E}_i$ is of positive characteristic then $n_i = 0$.

The following paragraph should replace the paragraph in the published version of this paper.

Let $\widehat{F}$ be of characteristic zero, and let $i \leqslant i_0$. If a primitive pth root of unity ζ_p equals $u^{\sigma_i - 1}$ with $u \in U_{\widehat{E}_i}$, then $N_{\widehat{E}_i/\widehat{E}_{i-1}}(\zeta_p) = 1$. Hence if $\zeta_p \notin \widehat{L} \setminus \widehat{F}$ then $n_i = 0$. We will call extensions L/F for which no primitive pth root belongs to $\widehat{L} \setminus \widehat{F}$ *regular*. In particular, every extension in positive characteristic is regular, and if ζ_p belongs to F, then L/F is regular. *For the rest of the paper we assume that L/F is a regular extension.*

DEFINITION 3'. If $n_i = 0$, set $A^{(i)} \in U_{\widehat{N(L/E_i)}}$ to be equal to 1. If $n_i > 0$, let $A^{(i)} \in U_{\widehat{N(L/E_i)}}$ be a lifting of α_i with the following restriction: $A^{(i)}_{\widehat{E}_{i+1}}$ is not a root of unity of order a power of p and (this condition can be satisfied by multiplying the $\widehat{E}_{i+1}$-component of $A^{(i)}$, if necessary, by $\gamma^{\sigma_{i+1}-1}$ where $\gamma \in U_{\widehat{E_{i+1}}}$ is sufficiently small).

LEMMA 3. *If $n_i > 0$, then $\beta_{i+1} = {A^{(i)}_{\widehat{E}_{i+1}}}^{p^{n_i}}$ belongs to X_{i+1}.*

In addition, if $n_i > 0$, then $A^{(i)}$ can be chosen such that $\beta_{i+1} \notin \langle \zeta_p \rangle X_{i+1}^p$.

Proof. Clearly $N_{\widehat{E}_{i+1}/\widehat{E}_i}\beta_{i+1} = 1$, so $\beta_{i+1} = \pi_{E_{i+1}}^{\rho-1} u^{\sigma_{i+1}-1}$ with $\rho \in \mathrm{Gal}(E_{i+1}/E_i)$. We need to show that $\rho = 1$.

Since $A^{(i)}_{\widehat{E}_i} \in U_{\widehat{E}_i}^{\sigma_i - 1}$ is a primitive root of unity of order a power of p, we deduce that $0 < s_i < e(E_i)/(p-1)$, hence $(s_i, p) = 1$. Note that if s_{i+1} is divisible by p, then from [FV, sect. 1 Ch. III] s_{i+1} must be equal to $pe(\widehat{E_i})/(p-1)$, and so $s_{i+1} > s_i$. It is well known, however, that since $s_i < s_{i+1}$ they have the same remainder modulo p, so s_{i+1} is prime to p, a contradiction. Thus, s_{i+1} is prime to p.

Let $\delta^{1-\varphi} = {A^{(i)}_{\widehat{E}_{i+1}}}^{p^{n_i - 1}}$. Then $\boldsymbol{N}(\rho) = (N_{\widehat{E_{i+1}}/\widehat{E}_i}\delta)^p \mod N_{E_{i+1}/E_i}U_{E_{i+1}}$. If $\varepsilon = N_{\widehat{E_{i+1}}/\widehat{E}_i}\delta \in E_i$, then $\boldsymbol{N}(\rho)$ belongs to $N_{E_{i+1}/E_i}U_{E_{i+1}}$, and hence $\rho = 1$. If $\varepsilon \notin E_i$, then $\varepsilon^p = a^p\omega$ where $a \in U_{E_i}$ and $\omega \in U_{E_i}$ is a p-primary element (the extension $E_i(\sqrt[p]{\omega})/E_i$ is unramified of degree p). If $\omega \in N_{E_{i+1}/E_i}U_{E_{i+1}}$, then $\boldsymbol{N}(\rho) = \varepsilon^p$ belongs to $N_{E_{i+1}/E_i}U_{E_{i+1}}$, and hence $\rho = 1$. If $\omega \notin N_{E_{i+1}/E_i}U_{E_{i+1}}$, then $s_{i+1} = pe(E_i)/(p-1)$, a contradiction. Thus, $\rho = 1$.

Since s_{i+1} is prime to p, ζ_p belongs to X_{i+1}. On the other hand, $\pi_{E_{i+1}}^{\sigma_{i+1}-1} \notin X_{i+1}$. If ${A^{(i)}_{\widehat{E}_{i+1}}}^{p} = \zeta_p^j x^p$ with $x \in X_{i+1}$, then, since $N_{\widehat{E}_{i+1}/\widehat{E}_i} A^{(i)}_{\widehat{E}_{i+1}} = \zeta_p$, we deduce that $A^{(i)}_{\widehat{E}_{i+1}} = x\zeta_{p^2}^k$ with k prime to p. Replace then $A^{(i)}$ with $A_1^{(i)}$ such that $A^{(i)}_{1\,\widehat{E}_{i+1}} = A^{(i)}_{\widehat{E}_{i+1}} \pi_{E_{i+1}}^{\sigma_{i+1}-1}$. If $A^{(i)p}_{1\,\widehat{E}_{i+1}}$ were in $\langle \zeta_p \rangle X_{i+1}$, then we would have $A^{(i)}_{1\,\widehat{E}_{i+1}} = y\zeta_{p^2}^l$ with l prime to p and $y \in X_{i+1}$. Therefore, $\pi_{E_{i+1}}^{\sigma_{i+1}-1} = \zeta_{p^2}^{l-k} y x^{-1}$. Applying $N_{\widehat{E}_{i+1}/\widehat{E}_i}$ we deduce that $\zeta_{p^2}^{l-k} \in \langle \zeta_p \rangle \leqslant X_{i+1}$ and so $\pi_{E_{i+1}}^{\sigma_{i+1}-1} \in X_{i+1}$, a contradiction. Thus, $A_1^{(i)}$ is the required element. □

From now one we denote by $A^{(i)}$ an element of $\widehat{N(L/F)}$ which can be chosen in accordance with the previous lemma.

We assume that if F is of characteristic 0 and contains a primitive pth root of unity, then L/F is of infinite degree.

DEFINITION 3". Let $\beta_{i,j}$, $j \in \mathbb{N}$ be free generators of Y_i which include β_i whenever β_i is defined (keeping in mind the right choice of β_i according to Lemma 3). Let $B^{(i,j)} \in U_{\widehat{N(L/E_i)}}$ be a lifting of $\beta_{i,j}$ (i.e. $B^{(i,j)}{}_{\widehat{E}_i} = \beta_{i,j}$), such that if $\beta_{i,j} = \beta_i$, then $B^{(i,j)}_{\widehat{E}_k} = B^{(i)}_{\widehat{E}_k} = A^{(i-1)^{p^{n_{i-1}}}}_{\widehat{E}_k}$ for $k \geqslant i$.

DEFINITION 4. Define a map $X_i \to U_{\widehat{N(L/E_i)}}$ by sending $\alpha_i^c \prod_j \beta_{i,j}^{c_j}$, where $0 \leqslant c \leqslant p^{n_i} - 1$, $c_j \in \mathbb{Z}_p$, to $A^{(i)^c} \prod_j B^{(i,j)^{c_j}}$. We get a map

$$f_i \colon U^{\sigma_i - 1}_{\widehat{E}_i} \to U_{\widehat{N(L/E_i)}} \to U_{\widehat{N(L/F)}}.$$

Note that $f_i(\alpha)_{\widehat{E}_i} = \alpha$.

DEFINITION 5. Denote by Z_i the image of f_i. Z_i depends on the choice of the lifting, but is unique up to an isomorphism. Let

$$Z_{L/F} = Z_{L/F}(\{E_i, f_i\}) = \left\{ \prod_i z^{(i)} : z^{(i)} \in Z_i \right\}$$

and

$$Y_{L/F} = \{ y \in U_{\widehat{N(L/F)}} : y^{1-\varphi} \in Z_{L/F} \}.$$

LEMMA 4. *The product of $z^{(i)}$ in the definition of $Z_{L/F}$ converges. $Z_{L/F}$ is a subgroup of $U^{\diamond}_{\widehat{N(L/F)}}$. The subgroup $Y_{L/F}$ contains $U_{N(L/F)}$.*

Proof. Let L/F be infinite. If $h_{L/F}$ is the Hasse–Herbrand function of L/F, then $h(L/F)^{-1}(s_i)$ tend to $+\infty$ when i tends to infinity [FV, Ch. III, sect. 5]. So, if $q(E_k|E_i)$ is the minimal real number such that $h_{E_k/E_i}(x) = x$ for $x \leqslant q(E_k|E_i)$, then $q(E_k|E_i)$, $k > i$, tends to infinity when i tends to infinity (see [FV, Ch.III,(5.2)]). Note that $v_{\widehat{E}_i}(z^{(i)}_{\widehat{E}_i} - 1) > s_i$, so from [FV, Ch.III, (5.5)] we deduce that

$$v_{\widehat{E}_i}((z^{(i)} - 1)_{\widehat{E}_i}) \geqslant \min((1 - p^{-1}) \min_{k>i} q(E_k|E_i), s_i).$$

Hence the product of $z^{(i)}$ converges.

Due to the previous definitions the subgroup of $U_{\widehat{N(L/F)}}$ generated by Z_i is equal to

$$\{ A^{(i)^c} \prod_j B^{(i,j)^{c_j}} : c, c_j \in \mathbb{Z}_p \}.$$

Since $A^{(i)^{p^{n_i} d}} = B^{(i+1)^d}$, $Z_{L/F}$ is a subgroup of $U_{\widehat{N(L/F)}}$. □

Denote by $U^1_{\widehat{N(L/F)}}$ the subgroup of elements whose $\widehat{F}$-component is 1.

The first assertion of the following theorem is a generalization of the fundamental exact sequence.

THEOREM 1. *The map*

$$\mathrm{Gal}(L/F) \to U^1_{\widehat{N(L/F)}}/Z_{L/F}, \quad \tau \mapsto X^{\tau-1}$$

is a bijection.

For every $(u_{\widehat{E_i}}) \in U^{\diamond}_{\widehat{N(L/F)}}$ *there is a unique automorphism* $\tau \in \mathrm{Gal}(L/F)$ *satisfying*

$$(u_{\widehat{E_i}})^{1-\varphi} \equiv X^{\tau-1} \bmod Z_{L/F}.$$

If $(u_{\widehat{E_i}}) \in Y_{L/F}$, *then* $\tau = 1$.

Proof. The first assertion certainly implies the second; and the proof of the second assertion below also verifies the first assertion of the theorem.

Assume that $u_{\widehat{E_{j-1}}} \in U_{E_{j-1}}$, $u_{\widehat{E_j}} \notin U_{E_j}$. Then $N_{\widehat{E_j}/\widehat{E_{j-1}}} u_{\widehat{E_j}}^{1-\varphi} = 1$, so from the fundamental exact sequence $u_{\widehat{E_j}}^{1-\varphi} = \pi_{E_j}^{\tau_j-1} w^{\sigma_j-1}$ with $\tau_j \in \mathrm{Gal}(E_j/E_{j-1})$, $w \in U_{\widehat{E_j}}$. Both τ_j and w^{σ_j-1} are uniquely determined by $(u_{\widehat{E_i}})$. Let $w^{(j)} = f_j(w^{\sigma_j-1})$.

Now assume that for $i > j$ we get

$$u_{\widehat{E_{i-1}}}^{1-\varphi} = \pi_{E_{i-1}}^{\tau_{i-1}-1} \prod_{j\leqslant k\leqslant i-1} w^{(k)}_{\widehat{E_{i-1}}}$$

with uniquely determined $\tau_{i-1} \in \mathrm{Gal}(E_{i-1}/F)$, $w^{(k)} \in f_k(U_{\widehat{E_k}}^{\sigma_k-1})$. We shall show that a similar statement holds for $u_{\widehat{E_i}}^{1-\varphi}$.

Let $\tau'_{i-1} \in \mathrm{Gal}(E_i/F)$ be an extension of τ_{i-1}. Then

$$N_{\widehat{E_i}/\widehat{E_{i-1}}} u_{\widehat{E_i}}^{1-\varphi} = N_{\widehat{E_i}/\widehat{E_{i-1}}} \big(\pi_{E_i}^{\tau'_{i-1}-1} \prod_{j\leqslant k\leqslant i-1} w^{(k)}_{\widehat{E_i}}\big),$$

so

$$u_{\widehat{E_i}}^{1-\varphi} = \varepsilon \pi_{E_i}^{\tau'_{i-1}-1} \prod_{j\leqslant k\leqslant i-1} w^{(k)}_{\widehat{E_i}}$$

with ε in the kernel of $N_{\widehat{E_i}/\widehat{E_{i-1}}}$. From the fundamental exact sequence we get $\varepsilon = \pi_{E_i}^{\sigma_i^a-1} v_i^{\sigma_i-1}$ with $v_i \in U_{\widehat{E_i}}$, $1 \leqslant a \leqslant |E_i : E_{i-1}|$. Write

$$\pi_{E_i}^{\tau'_{i-1}-1} \pi_{E_i}^{\sigma_i^a-1} = \pi_{E_i}^{\sigma_i^a\tau'_{i-1}-1} (\pi_{E_i}^{1-\tau'_{i-1}})^{\sigma_i^a-1}.$$

Put $\tau_i = \sigma_i^a \tau'_{i-1}|_{E_i}$, $w_i = v_i(\pi_{E_i}^{1-\tau'_{i-1}})^{1+\sigma_i+\dots+\sigma_i^{a-1}}$, $w^{(i)} = f_i(w_i^{\sigma_i-1})$. Then

$$u_{\widehat{E_i}}^{1-\varphi} = \pi_{E_i}^{\tau_i-1} \prod_{j\leqslant k\leqslant i} w^{(k)}_{\widehat{E_i}}.$$

It remains to show that τ_i, $w^{(i)}$ are uniquely determined. If $\rho_i \in \mathrm{Gal}(E_i/F)$, $v^{(i)} \in f_i(U_{\widehat{E_i}}^{\sigma_i-1})$ are such that

$$u_{\widehat{E_i}}^{1-\varphi} = \pi_{E_i}^{\rho_i-1} v^{(i)}_{\widehat{E_i}} \prod_{j\leqslant k\leqslant i-1} w^{(k)}_{\widehat{E_i}}$$

and $\tau_i|_{E_{i-1}} = \rho_i|_{E_{i-1}}$, then $\tau_i = \rho\rho_i$ with $\rho \in \mathrm{Gal}(E_i/E_{i-1})$ and

$$\pi_{E_i}^{\rho-1} = (\pi_{E_i}^{\rho_i-1})^{1-\rho} v_{\widehat{E}_i}^{(i)} / w_{\widehat{E}_i}^{(i)}.$$

Since $v_{\widehat{E}_i}^{(i)}, w_{\widehat{E}_i}^{(i)} \in U_{\widehat{E}_i}^{\sigma_i - 1}$, we deduce that the right hand side of the last equation belongs to $V(E_i|E_{i-1})$. Therefore, from the fundamental exact sequence $\rho = 1$. Consequently $v_{\widehat{E}_i}^{(i)} = w_{\widehat{E}_i}^{(i)}$ and $v^{(i)} = w^{(i)}$.

Thus, there is a unique automorphism $\tau \in \mathrm{Gal}(L/F)$, $\tau|_{E_i} = \tau_i$, satisfying

$$(u_{\widehat{E}_i})^{1-\varphi} = X^{\tau-1} z \quad \text{with } z \in Z_{L/F}.$$

If $(u_{\widehat{E}_i}) \in Y_{L/F}$, then from the uniqueness we get $\tau = 1$. □

COROLLARY. *Thus, we get the map*

$$\boldsymbol{H}_{L/F} \colon U^{\diamond}_{\widehat{N(L/F)}} \to \mathrm{Gal}(L/F)$$

defined by $\boldsymbol{H}_{L/F}((u_{\widehat{E}_i})) = \tau$. *The composition of* $\boldsymbol{N}_{L/F}$ *and* $\boldsymbol{H}_{L/F}$ *is the identity map of* $\mathrm{Gal}(L/F)$. ■

DEFINITION 6. Define the map

$$\mathcal{H}_{L/F} \colon U^{\diamond}_{\widehat{N(L/F)}} / Y_{L/F} \to \mathrm{Gal}(L/F)$$

by

$$\mathcal{H}_{L/F}((u_{\widehat{E}})) = \tau,$$

where τ is the unique automorphism satisfying $(u_{\widehat{E}})^{1-\varphi} \equiv X^{\tau-1} \bmod Z_{L/F}$.

Note that $U^{\diamond}_{\widehat{N(L/F)}} / Y_{L/F}$ is a direct product of a quotient group of the group of multiplicative representatives of the residue field of L, a cyclic group $\mathbb{Z}/p^a\mathbb{Z}$ and a countable free topological $\mathbb{Z}_p$-module.

LEMMA 5. $\mathcal{H}_{L/F}$ *is injective.*

Proof. If $(u_{\widehat{E}})^{1-\varphi} \in Z_{L/F}$, then $(u_{\widehat{E}}) \in Y_{L/F}$. □

THEOREM 2. *The map* $\mathcal{H}_{L/F}$ *and the map* $\mathcal{N}_{L/F} \colon \mathrm{Gal}(L/F) \to U^{\diamond}_{\widehat{N(L/F)}} / Y_{L/F}$ *induced by* $\boldsymbol{N}_{L/F}$ *are inverse bijections.*

Proof. From the definitions $\mathcal{H}_{L/F} \circ \mathcal{N}_{L/F} = \mathrm{id}$. Since $(u_{\widehat{E}_i})^{1-\varphi} = (u'_{\widehat{E}_i})^{1-\varphi} z$ with $z \in Z_{L/F}$ implies $(u_{\widehat{E}_i})(u'_{\widehat{E}_i})^{-1} \in Y_{L/F}$ we get $\mathcal{N}_{L/F} \circ \mathcal{H}_{L/F} = \mathrm{id}$. It remains to refer to Lemma 2. □

COROLLARY. $\mathrm{im}(\boldsymbol{N}_{L/F}) \cap Y_{L/F}/U_{N(L/F)} = (1)$ *and* $\mathrm{im}(\boldsymbol{N}_{L/F})$ *is a set of representatives of* $U^{\diamond}_{\widehat{N(L/F)}} / U_{N(L/F)}$ *modulo* $Y_{L/F}/U_{N(L/F)}$.

Proof. $\operatorname{im}(\boldsymbol{N}_{L/F}) \cap Y_{L/F}/U_{N(L/F)}$ coincides with the intersection of $\operatorname{im}(\boldsymbol{N}_{L/F})$ and the kernel of the epimorphism $U^{\diamond}_{\widehat{N(L/F)}}/U_{N(L/F)} \to U^{\diamond}_{\widehat{N(L/F)}}/Y_{L/F}$ which is trivial according to Theorem 2. The second assertion follows in the same way. □

REMARK 3. To the natural homomorphism $\lambda\colon \operatorname{Gal}(L/F) \to \operatorname{Aut} U^{\diamond}_{\widehat{N(L/F)}}/U_{N(L/F)}$, $\lambda(\sigma)x = \sigma x$, the 1-cocycle $\mathcal{N}_{L/F}$ associates a twisted monomorphism $\lambda_{\mathcal{N}}$ which acts from $\operatorname{Gal}(L/F)$ into the group $\operatorname{Aut}_S U^{\diamond}_{\widehat{N(L/F)}}/U_{N(L/F)}$ of automorphisms of $U^{\diamond}_{\widehat{N(L/F)}}/U_{N(L/F)}$ as a set defined by

$$\lambda_{\mathcal{N}}(\sigma)u = \boldsymbol{N}_{L/F}(\sigma)\sigma(u).$$

REMARK 4 (ON ABELIAN CLASS FIELD THEORY). To deduce abelian reciprocity map from Theorem 2 it suffices to prove that for an abelian extension L/F the natural epimorphism

$$U^{\diamond}_{\widehat{N(L/F)}}/Y_{L/F} \to U_F/N_{L/F}U_L, \qquad (u_{\widehat{E}}) \bmod Y_{L/F} \to u_{\widehat{F}} \bmod N_{L/F}U_L$$

is an isomorphism.

Assume that $(u_{\widehat{E}}) \in U^{\diamond}_{\widehat{N(L/F)}}$, $u_{\widehat{F}} \in N_{L/F}U_L$ and show that then $(u_{\widehat{E}}) \in Y_{L/F}$.

Let $(v_E) \in U_{N(L/F)}$ with $v_F = u_{\widehat{F}}$. Then $u_{\widehat{E}}v_E^{-1}$ belongs to the kernel of $N_{\widehat{E}/\widehat{F}}$, so from the fundamental sequence $u_{\widehat{E}}v_E^{-1} = \pi_E^{\sigma_E - 1}v$ with $v \in V(E/F)$. Then $u_{\widehat{E}}^{1-\varphi} \in V(E/F)$.

Furthermore, let by induction on i

$$u_{\widehat{E_{i-1}}}^{1-\varphi} = \prod_{j \leqslant k \leqslant i-1} w^{(k)}_{\widehat{E_{i-1}}}, \qquad w^{(k)} \in f_k(U_{\widehat{E_k}}^{\sigma_k - 1})$$

as in the proof of Theorem 1. Then

$$u_{\widehat{E_i}}^{1-\varphi} = \alpha^{\sigma_i - 1} \prod_{j \leqslant k \leqslant i-1} w^{(k)}_{\widehat{E_i}}$$

with some $\alpha \in \widehat{E_i}^*$. Since $u_{\widehat{E_i}}^{1-\varphi}, w^{(k)}_{\widehat{E_i}}$ belong to $V(E_i/F)$ and the extension E_i/F is abelian, from the fundamental exact sequence we deduce that $\alpha^{\sigma_i - 1} \in U_{\widehat{E_i}}^{\sigma_i - 1}$. Put $w^{(i)} = f_i(\alpha^{\sigma_i - 1})$, then

$$u_{\widehat{E_i}}^{1-\varphi} = \prod_{j \leqslant k \leqslant i} w^{(k)}_{\widehat{E_i}}.$$

Thus, $(u_{\widehat{E}}) \in Y_{L/F}$. In particular, $Y_{L/F}$ doesn't depend on the choice of E_i, f_i.

For a totally ramified Galois arithmetically profinite extension L/F denote by $U^1_{\widehat{N(L/F)}}$ the subgroup of $U^{\diamond}_{\widehat{N(L/F)}}$ of elements with the ground component 1. Then $\mathcal{H}_{L/F}(U^1_{\widehat{N(L/F)}}Y_{L/F}/Y_{L/F})$ is equal to the commutator subgroup $\operatorname{Gal}(L/F)'$ of $\operatorname{Gal}(L/F)$. The latter is mapped by $\boldsymbol{N}_{L/F}$ into a subset of $U^1_{\widehat{N(L/F)}}U_{N(L/F)}/U_{N(L/F)}$.

REMARK 5 (ON METABELIAN CLASS FIELD THEORY). Denote by q the cardinality of the residue field of F. Koch and de Shalit in [KdSh] constructed a metabelian class field theory which in particular describes totally ramified metabelian extensions of F (the commutator group of the commutator group is trivial) in terms of the following group (we indicate here only that part of their group $\mathfrak{G}(F,\varphi)$ which is relevant for totally ramified extensions)

$$\mathfrak{g}(F) = \{(u \in U_F, \xi(X) \in \mathbb{F}_p^{\rm sep}[[X]]^*) : \xi(X)^{\varphi-1} = \{u\}(X)/X\}$$

with a certain group structure. Here $\{u\}(X)$ is the residue series in $\mathbb{F}_p^{\rm sep}[[X]]^*$ of the endomorphism $[u](X) \in O_F[[X]]$ of the formal Lubin–Tate group corresponding to $\pi_F,\ q,\ u$.

Denote $R = F^{\rm ab} \cap F^{\varphi},\ M = R^{\rm ab} \cap F^{\varphi}$. Since R/F and $N(M,R/F)/N(R/F)$ are arithmetically profinite, the extension M/F is arithmetically profinite [W, 3.4.1].

Note that if $\tau = \boldsymbol{H}(u^{-1})$ is the automorphism of Gal(R/F) corresponding via abelian class field theory to u^{-1}, then the equation $\xi(X)^{\varphi-1} = \{u\}(X)/X$ can be interpreted as $(u_{\widehat{E}})^{1-\varphi} = (\pi_E)^{\tau-1}$ in the field of norms of R/F. This follows from the description of the field of norms construction, as, e.g., in [W, 3.2.5.1], since $(p-1)(q^{i+1}-1)/(p(q-1)^2q^{i-1}) \to (1-p^{-1})(1-q^{-1})^{-2}$. Hence $\mathfrak{g}(F)$ can be identified with the $U_F \times U_{N(R/F)}$-torsor $\mathfrak{n}(F)$ of all pairs $(u,(u_{\widehat{E}})) \in U_F \times U^{\diamond}_{\widehat{N(R/F)}}$ which satisfy the equation $(u_{\widehat{E}})^{1-\varphi} = (\pi_E)^{\boldsymbol{H}(u^{-1})-1}$.

According to Corollary of Theorem 2 every coset of $U^{\diamond}_{\widehat{N(M/F)}}$ modulo $Y_{M/F}$ has a unique representative in $\operatorname{im}(\boldsymbol{N}_{M/F})$. Send a coset with a representative

$$(u_{\widehat{Q}}) \in U^{\diamond}_{\widehat{N(M/F)}}$$

satisfying $(u_{\widehat{Q}})^{1-\varphi} = (\pi_Q)^{\tau-1}$ with $\tau \in$ Gal(M/F) to

$$\big(u_{\widehat{F}}^{-1}, (u_{\widehat{E}}) \in U^{\diamond}_{\widehat{N(R/F)}}\big).$$

It belongs to $\mathfrak{n}(F)$ by Remark 4. Thus, we get a map

$$g\colon U^{\diamond}_{\widehat{N(M/F)}}/Y_{M/F} \to \mathfrak{n}(F).$$

Now we construct an inverse map to g. For a pair $(u,(u_{\widehat{E}})) \in U_F \times U^{\diamond}_{\widehat{N(R/F)}}$ satisfying $(u_{\widehat{E}})^{1-\varphi} = (\pi_E)^{\tau-1}$ fix a finite subextension E/F of R/F. We claim that for every finite abelian subextension Q/E of M/E such that Q is normal over F there are unique $u_{\widehat{Q}} \in U_{\widehat{Q}}$ and $\tau_Q \in$ Gal(Q/F) satisfying

$$N_{\widehat{Q}/\widehat{E}} u_{\widehat{Q}} = u_{\widehat{E}}, \quad u_{\widehat{Q}}^{1-\varphi} = \pi_Q^{\tau_Q-1}, \quad \tau_Q|_E = \tau|_E.$$

Write $u_{\widehat{E}} = N_{\widehat{Q}/\widehat{E}} u$ with $u \in U_{\widehat{Q}}$ and observe that $N_{\widehat{Q}/\widehat{E}}(u^{1-\varphi}\pi_Q^{1-\tau'}) = 1$ for a lifting $\tau' \in$ Gal(Q/F) of τ. The group $V(Q/E)$ is $(\varphi-1)$-divisible, so $\pi_Q^{\sigma\tau'-1} = w^{\varphi-1}\pi_Q^{\tau'-1}\pi_Q^{\sigma-1}$ with $w \in V(Q/E)$. From the fundamental exact sequence for the abelian extension Q/E we get $u^{1-\varphi}\pi_Q^{1-\tau'} = \pi_Q^{\sigma-1}v^{\varphi-1}$ for some $\sigma \in$ Gal(Q/E), $v \in V(Q/E)$. Hence $u_{\widehat{Q}} = uvw^{-1}$ satisfies $N_{\widehat{Q}/\widehat{E}} u_{\widehat{Q}} = u_{\widehat{E}},\ u_{\widehat{Q}}^{1-\varphi} = \pi_Q^{\tau_Q-1}$, where $\tau_Q = \sigma\tau'$.

If ${u'_{\widehat{Q}}}^{1-\varphi} = \pi_Q^{\tau'_Q-1}$ and $N_{\widehat{Q}/\widehat{E}} u'_{\widehat{Q}} = u_{\widehat{E}}$, then $u_{\widehat{Q}}^{-1}u'_{\widehat{Q}}$ belongs to the kernel of $N_{\widehat{Q}/\widehat{E}}$, so from the fundamental sequence we deduce that $(\pi_Q^{\tau_Q})^{\tau'_Q\tau_Q^{-1}-1} = (u_{\widehat{Q}}^{-1}u'_{\widehat{Q}})^{1-\varphi} \in V(Q/E)$. Since Q/E is abelian, $\tau'_Q = \tau_Q,\ u'_{\widehat{Q}} = u_{\widehat{Q}}$.

Now let E_1/F be a subextension of an abelian extension E_2/F, let Q_1/E_1, Q_2/E_2 be abelian finite subextensions in M/F and let $Q_1 \subset Q_2$ be normal over F. Then

$$N_{\widehat{Q_2}/\widehat{Q_1}} u_{\widehat{Q_2}}, \tau_{Q_2}|_{Q_1},$$

where $u_{\widehat{Q_2}}$, τ_{Q_2} constructed for Q_2/E_2, satisfy the conditions for Q_1/E_1, therefore the uniqueness implies $N_{\widehat{Q_2}/\widehat{Q_1}} u_{\widehat{Q_2}} = u_{\widehat{Q_1}}$, $\tau_{Q_2}|_{Q_1} = \tau_{Q_1}$.

Hence the pair $(u, (u_{\widehat{E}})) \in U_F \times U^{\diamond}_{\widehat{N(R/F)}}$ satisfying $(u_{\widehat{E}})^{1-\varphi} = (\pi_E)^{\boldsymbol{H}(u^{-1})-1}$ uniquely determines $\tau_M \in \mathrm{Gal}(M/F)$ and $(u_{\widehat{Q}}) \in U^{\diamond}_{\widehat{N(M/F)}}$ satisfying $(u_{\widehat{Q}})^{1-\varphi} = (\pi_Q)^{\tau_M - 1}$.

Thus, we get the inverse map $h\colon \mathfrak{n}(F) \to U^{\diamond}_{\widehat{N(M/F)}}/Y_{M/F}$ to g.

Now it is easy to show that the reciprocity map

$$\mathfrak{n}(F) \to \mathrm{Gal}(M/F)$$

of [KdSh] coincides with $(\mathcal{H}_{M/F} \circ h)^{-1}$ and it associates τ_M^{-1} to $(u, (u_{\widehat{E}})) \in \mathfrak{n}(F)$. The map $(g \circ \mathcal{N}_{L/F})^{-1}$ is the inverse one.

Thus, without using Coleman's homomorphism and Lubin–Tate theory (employed in [KdSh]) one can deduce the metabelian reciprocity map as a partial case of $\mathcal{N}_{L/F}, \mathcal{H}_{L/F}$. Note that the group structure on $\mathfrak{g}(F)$ defined in [KdSh] corresponds to the group structure on $\mathrm{im}(\boldsymbol{N}_{M/F})$ discussed in Remark 2.

Remark 6. Similarly one can deduce the reciprocity map constructed by Gurevich [G] for extensions L/F for which the n-th derived group of the Galois group $\mathrm{Gal}(L/F)$ is trivial.

References

[D] B. Dwork, Norm residue symbol in local number fields, Abh. Math. Sem. Univ. Hamburg 22(1958), 180–190.

[F1] I. B. Fesenko, Class field theory of multidimensional local fields of characteristic 0, with the residue field of positive characteristic, St. Petersburg Math. J. 3(1992), 649–678.

[F2] I. B. Fesenko, Multidimensional local class field theory. II, St. Petersburg Math. J. 3(1992), 1103–1126.

[F3] I. B. Fesenko, Local class field theory: perfect residue field case, Russ. Acad. Scienc. Izvest. Math. 43(1994), 65–81.

[F4] I. Fesenko, Abelian local p-class field theory, Math. Ann. 301(1995), 561–586.

[F5] I. Fesenko, On general local reciprocity maps, J. reine angew. Math. 473(1996), 207–222.

[FV] I. B. Fesenko and S. V. Vostokov, Local Fields and Their Extensions, AMS, Providence, R.I. 1993.

[G] A. Gurevich, On generalization of metabelian local class field theory, preprint 1997.

[H1] M. Hazewinkel, Abelian extensions of local fields, Thesis, Amsterdam Univ., 1969.

[H2] M. Hazewinkel, Local class field theory is easy, Adv. Math. 18(1975), 148–181.

[I1] K. Iwasawa, Local Class Field Theory, Iwanami-Shoten, Tokyo, 1980.

[I2] K. Iwasawa, Local Class Field Theory, Oxford Univ. Press and Clarendon Press, New York and Oxford, 1986

[K] H. Koch, Local class field theory for metabelian extensions, In Proceed. of the 2nd Gauss Symposium. Conf. A: Mathematics and Theor. Physics (München, 1993), de Gruyter, Berlin, 1995, 287–300.

[KdSh] H. Koch and E. de Shalit, Metabelian local class field theory, J. reine angew. Math. 478(1996), 85–106.

[N1] J. Neukirch, Neubegründung der Klassenkörpertheorie, Math. Z. 186(1984), 557–574.

[N2] J. Neukirch, Class Field Theory, Springer, Berlin etc., 1986.

[Se] J.–P. Serre, Sur les corps locaux á corps residuel algébriquement clos, Bull. Soc. Math. France 89(1961), 105–154.

[W] J.–P. Wintenberger, Le corps des normes de certaines extensions infinies des corps locaux; applications, Ann. Sci. E.N.S., 4 série 16(1983), 59–89.

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