Several nonstandard remarks

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Hawking's Theory of Progress: Progress does not consist of replacing a theory that is wrong with one that is right. It consists of replacing a theory that is wrong with one that is more subtly wrong.

This text aims to present and discuss a number of situations in analysis, geometry, number theory and mathematical physics which can profit from developing their nonstandard description or interpretation (i.e. the study of the saturated theory) and then using it to prove standard results and/or establish standard theories.

Entirely nonstandard proofs are quite rare of much use in standard mathematics. One of the main obstructions is that it is hard to find a satisfactory standard shape of nonstandard (=hyper) structures. One of the great values of nonstandard mathematics is in serving as a "guiding star" and often offering a conceptually simple and elegant interpretation and generalization of classical structures. This provides a broader view of the standard theory and sometimes leads to new concrete standard results.

The text can be divided into five parts. The first short part comprises sections 1 and 2. Section 1 contains a calculation of the group of hyper integers, for which I was unable to find a reference. The calculation implies that the endomorphism ring of hyper integers is highly noncommutative, unlike the ring of endomorphisms of integers. This supplies one of several evidences towards explanation of the known observation that noncommutative methods might prove useful in the study of commutative structures in various parts of mathematics including number theory. Section 2 recalls a hyperfinite (or more generally hyperdiscrete) approximation principle, new applications/examples of which are proposed in later parts.

The second part consists of sections 3–7 where some of analytical and topological constructions are discussed from nonstandard mathematics point of view. Compact operators on a Hilbert space are interpreted in section 3 as infinitesimally small elements in the algebra of linear operators on the hyperspace, which leads to a number of questions on nonstandard interpretation (as standard images of hyper objects) of some important concepts of operator algebra theory in section 4. Two nonequivalent topologies on a hyper topological space are recalled in section 5. In section 6 to a "noncommutative space" we associate a hyper commutative one. It is very important that Morita equivalence is not compatible with the standard part (=shadow) map. This results in a remarkable phenomenon that hypercommutative theories descend to noncommutative theories at the classical

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level. In particular, this leads to many open questions about new commutative interpretations of various objects of noncommutative differential geometry, see section 7.

The third part consists of sections 8–9. Section 8 lists several of known points of interaction between noncommutative differential geometry and number theory, each of which possessing a suitable nonstandard commutative interpretation. Section 9 contains a research programme of hyper arithmetical approach towards realization of Manin's Alterstraum of real multiplication. It is based on using hyper lattices, hyper $\wp$-function and hyper division points of hyper elliptic curves. Quantum torii and quantum theta functions, known to have relevance for this programme, are expected to be images of hyper torii and hyper theta functions resp.

The fourth part comprises sections 10–13. Section 10 raises a question of existence of a hyper commutative description of a number of standard noncommutative theories. It also contains links of the current work to the theory of analytic Zariski structures by Zilber and geometric stability theory. Section 11 and 12 present several more examples of nonstandard interpretations of arithmetic algebraic geometry structures. Relations or applications of nonstandard mathematics to the central object of number theory – zeta functions are brought up in sections 12 and 13.

The last part discusses several applications in quantum physics, a very promising direction of further research.

The work concentrates on presenting main points rather than on already available technical details: those should appear elsewhere. Nonstandard mathematics forms a part of model theory, which might play in the future a very important role in explanation of a number of phenomena in number theory and quantum physics and their relation.

The origin of this work came from the study of translation invariant measures on generalized loop spaces with values in formal power series over reals and associated harmonic analysis with its applications in arithmetic geometry, see [Fe1, Remarks in sect. 4 and 13]. At later stages the work aimed to understand a series of papers at the intersection of noncommutative differential geometry and commutative number theory, and reasons underlying applications of the former to the latter. One of conclusions proposed in this work is a principle that whenever there is such an application, there is always at the background an entirely commutative description of the situation, and often in such situations it can be seen as the shadow image of a hyper commutative description.

Talks on parts of this work were delivered at seminars in Bonn, Oxford St. Petersburg and Cambridge, and I thank its participants for their remarks. Some of recent works ([B], [BS1–BS3], [T], [C]) are related to preliminary versions of this text. I am especially grateful to P. Cartier, Yu.I. Manin, M. Marcolli, B. Zilber and A.M. Vershik for their comments, explanations and suggestions.

The classical text on nonstandard mathematics is the Robinson book [Ro1]. For a short modern introduction into hyperobjects (mostly hyperreals) see [Gl] or shorter [AFHL, Ch. 1–3]. For recent reviews of various applications of nonstandard mathematics in analysis, statistics, differential equations, physics see [NAiP], [NA], [NAiA], [NAWM]; for some of results in nonstandard number theory 25 years ago see e.g. [RR], [Mc] and references therein. These are just some of vast literature on the subject. For an introduction into hyper categories see [BS1]. In sections 1, 3–4, 6–14 we mainly discuss situations not covered by the previous references.

1. Endomorphisms of hyper integers. We start with the basic object – abelian group $\mathbb{Z}$: its endomorphism ring is commutative canonically isomorphic to the ring of integers. The situation is drastically different when one calculates the endomorphism ring of the group ${}^*\mathbb{Z}$ of hyper integers. There is a natural epimorphism induced by ${}^*\mathbb{Z}/n \longrightarrow \mathbb{Z}/n$

$$ {}^*\mathbb{Z} \longrightarrow \prod \mathbb{Z}_p = \widehat{\mathbb{Z}} $$

whose kernel equals to $\cap_{n\in\mathbb{Z}\setminus\{0\}} n\,{}^*\mathbb{Z}$ and is an infinitely divisible group of uncountable cardinality. Hence

$$^*\mathbb{Z} \simeq \widehat{\mathbb{Z}} \times \mathbb{Q}^{\kappa},$$

with uncountable κ. This follows easily from the well known relation of $^*\mathbb{Z}$ and $\mathbb{Z}_p$, see for example [Ro5] or [Gl, Ch. 18]. Alternatively use the fact that the abelian group of integers is elementary equivalent to the saturated abelian group $\widehat{\mathbb{Z}} \times \mathbb{Q}^{\kappa}$, $^*\mathbb{Z}$ is saturated, and elementary equivalent saturated structures of the same cardinality are isomorphic, see e.g. [EF].

Thus, the endomorphism ring of $^*\mathbb{Z}$ is noncommutative. In particular, one can find two endomorphisms $A, B\colon {}^*\mathbb{Z} \longrightarrow {}^*\mathbb{Z}$ such that both A and B induce an identity operator on $\widehat{\mathbb{Z}}$ and $A \circ B - B \circ A$ is the identity operator on $^*\mathbb{Z} \setminus \widehat{\mathbb{Z}}$.

Note that by transfer the ring of internal endomorphisms of $^*\mathbb{Z}$ is the commutative ring $^*\mathbb{Z}$. See also section 10.

If one wants to get a hyperfinite instead of hyperdiscrete group, then choose $\omega \in {}^*\mathbb{N} \setminus \mathbb{N}$. There are two injective homomorphisms

$$\mathbb{Z} \longrightarrow {}^*\mathbb{Z}/\omega^*\mathbb{Z} \longrightarrow {}^*\mathbb{Z},$$

the last is not canonical and follows from a similar to the previous description $^*\mathbb{Z}/\omega^*\mathbb{Z} \simeq \widehat{\mathbb{Z}} \times \mathbb{Q}^{\kappa}$. The middle object is hyperfinite.

2. Hyperfinite approximation principle. One of the most important methodological tools to use nonstandard mathematics is the hyperfinite (more generally, hyperdiscrete) approximation: for a usual object X try to find a hyperfinite (or hyperdiscrete) object Y and morphisms

$$X \longrightarrow Y \longrightarrow {}^*X,$$

with $X \longrightarrow {}^*X$ being the diagonal embedding. See e.g. [Gl, Ch. 19]. By transfer the hyperobject $^*X$ has many properties similar to X, inducing those on Y and using the fact that Y has many properties similar to finite objects, one can then attempt to interpret various classical constructions on X.

For example, the existence of Haar measure on locally compact abelian groups can be established in the shortest way by inducing it from the hypercounting measure on a covering hyperfinite abelian group (e.g. [NA], [NAiA], [Gr1]); the same is also true for the Fourier transform [Gr2]. Hyperfinite measure spaces theory due to works of Loeb and Anderson has found many applications in measure theory. Notice that in the hypertheory one often works with approximate morphisms and functors (modulo the halo), e.g. [Gr2], [YO].

A generalization of Haar measure on the additive group of higher local fields [Fe1] can be viewed as induced from the counting measure on hyperhyperfinite abelian groups (or hyper Haar measure on hyper locally compact groups).

For more examples see the following sections.

3. Compact operators as infinitesimally small elements. For a Hilbert space X such a hyperfinite space Y does exist, e.g. [AFHL, Ch. 3]. Compact operators on X are restrictions of hyperfinite operators on Y which enjoy the additional property: they map finite points to nearstandard, e.g. [AFHL, 2.3]. We have algebra homomorphisms

$$\mathcal{L}(X) \longrightarrow \mathcal{L}({}^*X), \quad r\colon \mathcal{L}({}^*X) \longrightarrow \mathcal{L}(Y),$$

the second is given by $A \mapsto P \circ A\colon Y \longrightarrow Y$ where $P\colon Y \longrightarrow {}^*X$ is a projection. In particular, the algebra $\mathcal{K}(X)$ of compact operators on X can be viewed as a summand of the hyperfinite algebra $\mathcal{L}(Y)$ of linear operators on Y. The space $\mathcal{K}(X)$ contains the subspace S generated by the images of

$\mathcal{L}(Z)$, Z a finite subspace, and $\mathcal{K}(X)$ is the halo of S with respect to the hypernorm on $\mathcal{L}(Y)$. So the compact operators do play the role of infinitesimal objects. Compare this with the psychologically different point of view of Connes in [Co4, VIII] where the two classes are rather contrasted each other (and note that the Dixmier trace has also a natural nonstandard interpretation, see [Ro10]).

4. Nonstandard interpretation of operator algebras. A number of results on stellar and especially operator algebras (which are in a sense algebraic objects) is likely to have a nice (and hopefully useful) interpretation in the language of internal algebras of hyper finite spaces, with possibility to have a simple conceptual picture of what is really going on in existing often technically complicated classical proofs.

For example, the Morita equivalence studied by Rieffel [Ri1–Ri3] for stellar algebras $\mathcal{A}, \mathcal{B}$, which in the simplest case (if A, B have countable approximate identities, c.f. [Ri1]) is equivalent to stable isomorphism $\mathcal{A} \otimes \mathcal{K}(H) \simeq \mathcal{B} \otimes \mathcal{K}(H)$ for a separable Hilbert space H, must have a very simple interpretation in $\mathcal{L}(Y)$ (we view $\mathcal{A}, \mathcal{B}$ inside $\mathcal{L}(X)$).

Existence of a nonstandard interpretation would agree with the completeness of such algebras, and would correspond to the well known fact that the shadow image of an internal subset in a Hausdorff space is closed. From this point of view it seems more likely that one really needs completed versions of objects like cross algebras, see the last pages of [Mn2], as opposite to incomplete ones, even for applications to algebraic geometry and number theory. See also section 6.

5. Two topologies of a hyper space. Recall that there are various complications about topologies of the hyper space $^*X$ for a topological space X with topology τ: this hyper space does not have a canonical topology, and there are at least two important topologies on it: one called S-topology with basis of fundamental neighbourhoods generated by $^*U$, U runs through open subsets in X, and the other called Q-topology (i.e. $^*\tau$-topology) with basis of neighbourhoods consisting of $^*T$ where T is the set of a fundamental neighbourhoods in τ. (As usual, we always assume appropriate saturation level for the hyperuniverses). The space $^*X$ is compact with respect to the S-topology, and normally the space $^*X$ is not Hausdorff with respect to the S-topology. The restriction of the S-topology to X coincides with τ. The space $^*X$ is Hausdorff with respect to the Q-topology, and the restriction of the Q-topology to X normally does not coincide with τ.

The shadow map from nearstandard point of $^*X$ to X is defined using the S-topology, and hence is not quite compatible with the Q-topology. This makes many applications or constructions of nonstandard mathematics quite cumbersome, since the Q-topology is more natural in many issues.

6. Hyper commutative space associated to a noncommutative space. When the quotient space X/Y of a metric topological space X is not Hausdorff, it is sometimes called a "noncommutative space". In the case of $X/\mathbb{Z}$ some of its invariants are studied by working with the noncommutative algebra of functions which is an appropriate completion of the cross product of continuous or smooth functions on X with the action of $\mathbb{Z}$. A central initial example is the noncommutative algebra of functions associated to degenerate torus $\mathbb{C}/T_\theta$ where $T_\theta = \mathbb{Z} + \theta\mathbb{Z} \subset \mathbb{R}$, $\theta \in \mathbb{R} \setminus \mathbb{Q}$. The torus splits into the product of $\mathbb{R}$ and $\mathbb{R}/T_\theta$ and the latter can be viewed as the quotient of the compact space $\mathbb{R}/\mathbb{Z}$ by the induced via multiplication by θ action of integers. Via the exponential map the torus is isomorphic to $\mathbb{C}^\times/\langle \exp(2\pi i\theta)\rangle$ and the corresponding stellar or operator algebra is the appropriate completion of the algebra generated over $\mathbb{C}$ by x, y, x^{-1}, y^{-1} satisfying $xy = \exp(2\pi i\theta)\, yx$.

Now let θ_n be a sequence of complex numbers with non-zero imaginary part such that $\theta_n \to \theta$. It is known that noncommutative objects can be viewed as degenerate objects, as certain imprecise limits of commutative objects (see below section 8). In the particular case of the previous paragraph one can easily give more mathematical meaning to this. Let Θ be the equivalence class of (θ_n) in $^*\mathbb{C}$. Consider the stellar or operator algebra of functions on the hyper commutative torus $T_\Theta = {}^*\mathbb{C}/({}^*\mathbb{Z} + \Theta\,{}^*\mathbb{Z})$.

The stellar algebra is Morita equivalent to the algebra of internal continuous ${}^*\mathbb{C}$-valued functions on T_Θ. It is generated by X, Y satisfying $XY = \exp(2\pi i\Theta)\,YX$. The shadow image of this algebra is the noncommutative algebra of functions of T_θ (note that the Morita equivalence is not compatible with the shadow map). So we can view the function algebra of the noncommutative object T_θ as a quotient of the hyper function algebra of the hyper commutative T_Θ. For the latter one gets via transfer principle analogues of many properties of commutative torii, and then using the shadow map one can descend those to properties of the quantum torus, including its invariants supplied by operator algebras theory.

One has a similar description if S is a dense subspace and subgroup of a metric topological space and group T such that, as above, S is a "limit" of S_n and T/S_n are Hausdorff.

Denote $G = {}^*\mathbb{Z}\,/\omega{}^*\mathbb{Z}$ for $\omega \in {}^*\mathbb{N} \setminus \mathbb{N}$. This is a hyperfinite group and there are two injective homomorphisms $\mathbb{Z} \longrightarrow G \longrightarrow {}^*\mathbb{Z}$, the last is not canonical and follows from a similar to section 1 description $G \simeq \mathbb{Q}^\kappa \times \widehat{\mathbb{Z}}$. For a metric topological space X with an α-action of integers, the topological space ${}^*X$ has an action of hyperintegers. Endow ${}^*X$ with the Q-topology and consider the operator algebra $\mathcal{A}({}^*X)$ generated by (internally) continuous functions ${}^*X \longrightarrow {}^*\mathbb{C}$.

Now notice that the space ${}^*X/{}^*\mathbb{Z}$ and hence ${}^*X/G$ are Hausdorff. Thus one can use the "covering" ${}^*X/G \longrightarrow X/\mathbb{Z}$ (more precisely, the shadow map applied to the algebra of functions on ${}^*X/G$) to study refined invariants of the "noncommutative space" $X/\mathbb{Z}$ (i.e. invariants of the associated noncommutative algebra) by means of hyper classical invariants of the commutative space ${}^*X/G$. In particular, we have $(\mathcal{A}({}^*X)^G)'' \simeq \mathcal{A}({}^*X) \times_\alpha G$ and the stellar algebra of ${}^*X/G$ is Morita equivalent to the cross product of the stellar algebra of ${}^*X$ and G. Both assertions follow from the corresponding properties for finite groups, by transfer. The hyper object $\mathcal{A}({}^*X) \times_\alpha G$ behaves "better" than the original standard object, and it can be used to reinterpet well known results about the operator cross algebra $\mathcal{A}(X) \times_\alpha \mathbb{Z}$.

7. Nonstandard interpretation of elements of noncommutative differential geometry. It would be interesting to investigate how much geometry of noncommutative spaces ([Co1,2] and references therein) can be interpreted as appropriate standard images of geometry of appropriate hyper commutative spaces. Notice that the periodic cyclic homology and KK-constructions can be defined as analogous to infinitesimal homology in algebraic geometry (see [Cu] and references there), which in particular gives a nice simple interpretation of the bivariant Chern character. On the other hand, the infinitesimal homology can be viewed as a standard image of a nonstandard homology; see also section 11 for nonstandard interpretations of direct and inverse limits constructions. Of course, one of the key problems is to obtain a hyper interpretation of the spectral triple in noncommutative geometry.

8. Nonstandard interpretations of interactions between noncommutative differential geometry and number theory. Turning to some of existing applications of noncommutative differential geometry to number theory, one can ask to which extent nonstandard interpretations are useful there. For example, both deformations and representation theory of noncommutative objects are supposed to have lifts to hyper categories, and possibly be closely linked up there.

The attitude to view noncommutative spaces as boundaries, degenerate objects as sometimes belonging to the boundary of an appropriate modular space, and the boundary being "a bridge to the world of noncommutative differential geometry" proved to be fruitful [So], [SV], [MM1]. This is also related to the holographic principle whose importance in quantum field theory [O] and mathematics [MM2] becomes more and more evident.

When one works with boundaries, compactification issues come immediately into play. It is well known that nonstandard constructions often provide elegant ways to compactify spaces, c.f. [Ro8] and a number of later works. A compactification of a space S can be found (sometimes in a "natural" way) inside the space $^*S$. Viewing $PGL_2(\mathbb{Z}) \setminus (\mathbb{P}^1(\mathbb{R}) \setminus \mathbb{P}^1(\mathbb{Q}))$ as an invisible part of the modular curve [MM1], averages and limits of modular symbols [MM1],[Mr], all are compatible with the nonstandard point of view.

9. A new approach to real multiplication. A programme to develop a theory of real multiplication using new (for number theory) structures was recently raised by Manin [Mn3]. His own approach advocates the usefulness of noncommutative (quantum) torii and noncommutative (quantum) theta functions. Here we describe another approach to real multiplication based on using appropriate hyper objects. One of advantages of this approach is that it includes new integral structures playing a central role; those integral structures seem to be very difficult to obtain using quantum objects of [Mn3].

The torii in section 6 have ring structure if the lattice is generated by a quadratic irrationality. Let K_n be a sequence of quadratic imaginary number fields, and let $\theta_n = a_n + b_n \in K_n$, $a_n \in \mathbb{Q}$, $b_n \in i\mathbb{R} \setminus \{0\}$ be such that a_n tend to a quadratic real irrationality θ, lying in a real quadratic field F, and b_n tend to 0. Form a hypercomplex number Θ corresponding to (θ_n), and consider a hyper torus

$$T_\Theta = {}^*\mathbb{C}/L_\Theta, \quad L_\Theta = {}^*\mathbb{Z} + \Theta {}^*\mathbb{Z}\,.$$

The analytic parametrization in the basic theory of elliptic curves has its hyper analogue: define the hyper Weierstraß function

$$\wp_\Theta(z) = \frac{1}{z^2} + {}^*\!\!\sum_{l \in L_\Theta \setminus \{0\}} \left(\frac{1}{(z-l)^2} - \frac{1}{l^2} \right)$$

which for z being the equivalence class of (z_n), $z_n \in \mathbb{C}$, is the equivalence class of $(\wp_{\theta_n}(z_n))$. Consider a hyper elliptic (not in the standard sense of this combination of adjectives) curve E_Θ over $^*\mathbb{C}$ whose points are $(\wp_\Theta(z), \wp'_\Theta(z))$, $z \in T_\Theta$.

Via the transfer principle abelian extensions of the hyperfield $\mathcal{K}$ corresponding to the sequence (K_n) are explicitly described by hyper complex multiplication theory. The Galois group of $F^{\rm ab}/F$ is isomorphic to a quotient of $\mathcal{K}^{\rm ab}/\mathcal{K}$ and making this explicit would give an explicit description of the maximal abelian extension of F, i.e. what could be called a theory of real multiplication. Various standard images of hyper objects in the hyper complex multiplication theory associated to $\mathcal{K}$ and E_Θ, or their modifications, supply new analytic, geometric and arithmetic structures (including, importantly, various integral structures) for the theory of real multiplication. Some of them can also be used for an interpretation of noncommutative structures in [Mn3], e.g. quantum theta functions (e.g. [Mn2], [Mn3, sect. 2], [Sc]) as standard images of hyper theta functions. For further development see [T].

10. *Hyper commutative versus noncommutative.* In the previous case of nonstandard extension of interaction of noncommutative geometry and number theory the noncommutative description is closely related to a nonstandard commutative one. *How many more situations with commutative hyper objects at the background can be detected among other applications of noncommutative differential geometry, at least to number theory?* The material of section 1 might be the starting point of explanation why one gets noncommutative standard objects from commutative hyper objects.

There are important links between nonstandard mathematics and a part of model theory which support each other. The theory and technique of analytic Zariski structures developed by Zilber [Z1] leads to a good understanding (elimination of quantifiers in a geometric language) of limit objects; elimination of quantifiers is equivalent to understanding of automorphisms of a saturated model. It is conjectured in [Z2] that an expanded structure $(\mathbb{C}, +, \times, G)$ on real curve $G_\theta =$

$\exp((1+i)\mathbb{R})\exp(\theta 2\pi i\mathbb{Z})$ is superstable if and only if θ is a real quadratic irrationality. Assuming Shanuel's conjecture it is shown that the structure is indeed superstable and the indeterministic real curve G_θ is definable. If θ is rational, the ordinary real curve G_θ is not stable.

11. More instances of hyper discretization.

(1) Viewing local fields as subquotients of a hyper global field one can try to systematically reformulate in the hyper language and then further develop appropriate classical local theories in number theory and algebraic geometry (e.g. [Ro7–Ro9]). The list of such theories may include infinite dimensional algebraic geometry (see e.g. [B]), resolution of singularities, and many other theories in which ind and pro objects play an essential role.

(2) Applying the same general mechanism one obtains a discretization and algebraization of analytic and topological theories. Of course, nonstandard analysis is the first primary example of this trend in modern mathematics. In particular, in the case where such a theory exists over archimedean fields only, one can try to extend the lower level hyper discrete theory to an arbitrary base, thus hoping to obtain a reasonable analogue over other fields.

For example, objects in complex algebraic geometry can be viewed as coming from hyper discrete algebraic geometry over ${}^*\mathbb{Q}(i)$.

(3) By gluing lifts of objects from local theories over different places of a global object to hyper level, one might expect to have a unified theory in which the difference between archimedean and nonarchimedean places disappears. Would this lead to a conceptually better presentation of Arakelov's theory? Recent results [CM] on relations of the latter and noncommutative differential geometry, which are relied on Manin's interpretation of hyperbolic geometry as a sort of archimedean Arakelov geometry [Mn1], are well suitable for a nonstandard interpretation as well.

(4) For the study of hyper étale cohomology theory see the recent work [BS2]. This theory has applications to l-adic cohomology, when one uses ultraproducts of finite fields for the coefficients of the cohomology. The latter object is quite useful, as was already demonstrated in [AK], and many other papers, see e.g. [FJ].

It is presumably clear that the list of instances can be extended more and more.

12. Hyper global objects and adelic objects. In nonstandard arithmetic one can interpret the adele and idele groups using hyper fractional ideals in hyper algebraic number fields [Ro2–Ro6]; in particular, $\mathbb{A}_{\mathbb{Q}}$ can be viewed as a subquotient of ${}^*\mathbb{Q}$ and the latter is sometimes easier to work with. This, together with the previous remarks, leads to the natural question *if one could derive a hyper approach to one dimensional and higher dimensional class field theory?*

Interpreting zeros of the Riemann zeta function (c.f. [Co3]) it might be useful to work not only with the space $\mathbb{A}/\mathbb{Q}^\times$ but with much larger and more interesting (and easier to study) space ${}^*\mathbb{Q}/\mathbb{Q}^\times$ as well.

Remarks in the previous section and standard images of hyper global objects and of function spaces on them are closely related to "q-world" objects in the approach by Haran [H]. "Quantum" q-objects which specialize both to p-adic and archimedean objects can be viewed as standard images of hyper rational objects. For example [H] put

$$\zeta_q(s) = \prod_{n\geqslant 0}(1-q^{s+n})^{-1},$$

then

$$\zeta_q(s) \to \zeta_p(s) = 1/(1-p^{-s}), \quad \text{if } q,s\to 0 \text{ and } q^s \to p^{-s}$$

for prime p, and

$$\zeta_q(s) \to \Gamma(s) = \zeta_\infty(s)\pi^{s/2}, \quad \text{if } q \to 1 \text{ and } (1-q^s)/(1-q) \to s.$$

For similar examples see [BF, sect. 14]. See also section 14.

Several constructions in Haran's theory can be viewed as shadow images of hyperdiscrete objects, and it would be very interesting to systematically investigate this.

13. Nonstandard approach to complex analysis and zeta functions. Not so much is known about nonstandard complex analysis (just [Ro1, Ch.VI], [Fru], [Ca]), unlike nonstandard real analysis, topology and metric spaces theory. Investigation in this area and the study of hyper analytic number theory might be very useful for a conceptual interpretation of some of central objects in analytic number theory including zeta functions. This study will undoubtfully be very useful for the work sketched in section 9.

In relation to zeta functions, a "hyper dream" is *whether various types of quite different zeta functions in number theory (complex valued and p-adic valued) could originate from a single source?* In the case of one dimensional theory such a source could be the hyper zeta functions of hyper A-fields K (say $\zeta_{{}^*K}({}^*s)$). It would be then this source from which zeta values drop as anticipated by Kato in [K]. For a first work on hyper zeta functions see [C].

14. Relations with physics. Hyperdiscrete constructions very well correspond to the way physicists argue in quantum physics: Combination of both discrete and continuous properties in hyperdiscrete objects is extremely promising for applications in mathematical physics: hyper discrete objects are ideally suited to describe the familiar type of wave–particle behaviour in physics through shadow images of nonstandard objects.

14.1. Divergent integrals ubiquitous in field theories can be naturally viewed as hypercomplex unlimited numbers, and various renormalization procedures could have enlightening nonstandard interpretations. In particular, nonrigorous physical constructions could be given mathematically sound justification. It is very surprising that almost nothing has been done in this direction. For first steps in nonstandard interpretations of aspects of quantum physic theories see e.g. [AFHL], [Y] and references therein.

14.2. Much of what is known about quantum field theory comes from perturbation theory and applications of Feynman diagrams for calculation of scattering amplitudes. Recall Feynman "never felt order by order was anything but an approximation to the path integral" [S, p.460]. The Feynman path integral is extremely difficult to give a mathematically sound theory, even though there are many attempts to produce such; each of them has some drawbacks (see [Fe2, sect. 18]). In particular, the value of the integral has a rigorous mathematical meaning as a hypercomplex number, manipulations with which do produce standard complex numbers seen in the recipes of Feynman, see e.g. [AFHL]. However, several attempts to construct and apply a nonstandard theory of the path integral (e.g. [L1,L2], [N1,N2]) have not yet led to a successful and practically useful general theory. Recall that Wiener measure (often used in mathematical approaches to the path integral) can be viewed as the Loeb measure associated to hyper random discrete walk, e.g. [NA], [NAiA], [NAiP]. As a generalization of the measure on two dimensional local fields [Fe1] which itself is induced from a hyper Haar measure, one might hope to develop a rigorous approach to a *shift invariant* measure behind the Feynman integral. See [Fe2] for the measure and integration on algebraic loop spaces.

14.3. A string with infinitesimally small diameter can be viewed inside the halo of a standard path, and the path is its shadow. It is interesting to see if (parts of) string theory could be viewed as shadow images of hyper two dimensional quantum field theory, by passing from hypercomplex numbers to power series over $\mathbb{C}$ in which a variable corresponds to an infinitesimally small positive element, and then working with the theory which takes values in formal power series in X and at the last stage replacing X with the constant $\sqrt{\alpha'}$ as in the string theory (for a similar approach see [Fe2]).

14.4. Space–time structure. As explained in the number of examples in sections 3,4,6 various parts of operator theory and noncommutative differential geometry can be viewed as shadow images of appropriate hyper constructions. Noncommutative differential geometry seems to find more and more relations and applications in string theory. So the physicists opinion "One of the great advantages of noncommutative geometry is precisely to put on the same ground the discrete and the continuous cases" [I, p.245] is entirely compatible with the previous statement. In relation to section 10 we mention an important map in string theory, called the Seiberg–Witten map [SW]. It relates a noncommutative gauge field with an ordinary gauge field in string theory. Observe that from the physicist point of view [SW, 6.4] focusing on the full algebra of operators acting on an open string leads naturally to finitely generated projective modules over a commutative torus, and the noncommutative torus appears when one "attempts to focus attention on just one end of the open string". Little seems to be known about analogues of this map in pure mathematics; those could be closely related to links between noncommutative descriptions and standard images of hyper commutative descriptions of the objects in the previous sections.

14.5. From the point of view of the previous material one can refine Riemann's words in [R]: "... bei einer discreten Mannigfaltigkeit das Princip der Massverhältnisse schon in dem Begriffe dieser Mannigfaltigkeit enthalten ist, bei einer stetigen aber anders woher hinzukommen muss. Es muss also entweder das dem Raume zu Grunde liegende Wirkliche eine discrete Mannigfaltigkeit bilden, oder der Grund der Massverhältnisse ausserhalb, in darauf wirkenden bindenen Kräften, gesucht werden." – if the continuous manifold is actually given as an image of a hyperdiscrete manifold, then it has the natural ground of its metric induced from the natural hyperdiscrete one.

In connection with Dyson's words "the marriage between mathematics and physics, which was so enormously fruitful in past centuries, has recently ended in divorce" [D], is it overoptimistic to ask: would in the future model theory help to recouple mathematics and parts of theoretical physics?

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