Adelic geometry and analysis on regular models of elliptic curves over global fields and their zeta functions

Ivan Fesenko

One-dimensional zeta function

Let k be a global field
(number field or function field of a curve over finite field)

The Euler product description of the *zeta function*

$$\zeta_k(s) = \prod_{\mathfrak{p}} (1 - |k(\mathfrak{p})|^{-s})^{-1} = \sum_{n \geq 1} \frac{a_n}{n^s}$$

The completed zeta function

$$\hat{\zeta}_k(s) = \zeta_k(s)\Gamma_k(s)$$

has an integral representation which in its adelic form is

$$\int_{\mathbb{A}_k^\times} f(x)|x|^s d\mu_{\mathbb{A}_k^\times}(x)$$

where f is a Bruhat-Schwarz function and $|\ |$ is the module function on ideles

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Using adelic duality and Fourier transform (which descends to the Gauss–Cauchy–Poisson summation formula) and rephrasing one of the Riemann computations, one gets

$$\hat{\zeta}_k(s) = \xi(f, s) + \xi(\hat{f}, 1 - s) + \omega(f, s),$$

where $\hat{f}$ is the transform of f, with the entire function $\xi(f, s)$.

The last boundary term can be written as

$$\omega(f, s) = \int_{N_-} \int_{\mathbb{A}_k^1/k^\times} \int_{\partial k^\times} \left(-f(n\gamma\beta)n^s + \hat{f}(n^{-1}\gamma\beta)n^{s-1}\right) d\mu(\beta) d\mu(\gamma) d\mu(n)$$

The weak boundary $\partial k^\times = k \setminus k^\times$ is just one point 0 and

$$\int_{\partial k^\times} \left(-f(n\gamma\beta)n^s + \hat{f}(n^{-1}\gamma\beta)n^{s-1}\right) d\mu(\beta) = -f(0)n^s + \hat{f}(0)n^{s-1}$$

whose integral over N_- ($= (0, 1)$ in characteristic 0) is a rational function of n symmetric with respect to $f \to \hat{f}, s \to 1 - s$

This 1d adelic method also proves the compactness of $\mathbb{A}_k^1/k^\times$, i.e. the finiteness of the class number, and easily implies Dirichlet's unit theorem.

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Dimension two

The zeta function of a scheme X of finite type over $\mathrm{Spec}(\mathbb{Z})$

$$\zeta_X(s) = \prod_{x \in X_0} (1 - |k(x)|^{-s})^{-1},$$

x runs through closed points of X, $k(x)$ is the finite residue field of x.

The zeta function $\zeta_X(s)$ factorizes into the product of some auxiliary factors and several L-factors or their inverses, which have an interpretation in terms of the action of the Frobenius endomorphism on cohomology groups associated to X in positive characteristic and as the product of such factors for all finite primes in characteristic zero.

Broadly speaking, the Langlands programme expects each of the L-factors to be equal to an appropriate L-function of an automorphic representation and hence have a meromorphic extension and satisfy a functional equation. The study of L-functions uses generally noncommutative or local (p-adic methods).

When the function field of X is of characteristic zero and X is two- or higher dimensional, very little is understood about $\zeta_X(s)$.

One can compare the zeta function to a molecule and its L-factors to atoms. Higher dimensional adelic analysis studies molecules rather than atoms which are the object for the Langlands programme.

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Dimension two

Here is a central open example in dimension two.
Let E be an elliptic curve over a global field k, and let $\mathcal{E}$ be a regular model, $\mathcal{E} \to B$ proper flat, where B is the spectrum of the ring of integers of k or a proper smooth curve over a finite field with function field k.

Then

$$\zeta_{\mathcal{E}}(s) = n_{\mathcal{E}}(s)\zeta_E(s), \qquad \zeta_E(s) = \frac{\zeta_k(s)\zeta_k(s-1)}{L_E(s)}$$

where

$$n_{\mathcal{E}}(s) = \prod_{b \in B_0, 1 \le i \le n_b} (1 - |k(b)|^{n_{i,b}(1-s)})^{-1}$$

is the product of zeta functions of affine lines over finite fields, $n_b + 1$ is the number of irreducible componens of the fibre $\mathcal{E}_b$,
$n_{i,b}$ are positive integers such that $1 + \sum_{1 \le i \le n_b} n_{i,b}$ equals the number m_b of irreducible components in the geometric fibre of $\mathcal{E}$ over b.

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Dimension two

The function $\zeta_E(s)$ was invented by Hasse and is sometimes called the Hasse–Weil zeta function of E, it does not depend on the choice of a model $\mathcal{E}$.

If the class number of k is 1 then $\zeta_E(s)$ is the zeta function of the minimal Weierstrass model of E.

The numerator of $\zeta_E(s)$ is the product of the zeta functions in dimension one. Its denominator is the L-function of E which was originally defined as the quotient $\zeta_k(s)\zeta_k(s-1)/\zeta_E(s)$ so one does not need to use/remember the bad reduction factors of the L-function.

The previous work in arithmetic of elliptic curves studies the L-function of E and not the zeta function of $\mathcal{E}$; and there is always some restriction on the algebraic number field k. The famous Wiles theorem and its further extensions imply meromorphic continuation and functional equation of the zeta function of $\mathcal{E}$ when k is a (imaginary quadratic extension of a) totally real field, but this method cannot be extended to the general case of k.

2d adelic analysis studies the zeta function $\zeta_{\mathcal{E}}$ directly, using commutative two-dimensional methods which universally work over any ground field k. The Galois group at the background is $\mathrm{Gal}(K^{\mathrm{ab}}/K)$ where K is a 2d global field, the field of function of $\mathcal{E}$. If E is given by an irreducible polynomial $y^2 - f(x) \in k[x,y]$ then K is the fraction field of $k[x,y]/(y^2 - f(x))$, and its transcendence degree over k is 1.

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2d adelic analysis

Aims of 2d adelic analysis in the case of arithmetic surfaces $\mathcal{E}$:
understand $\zeta_{\mathcal{E}}$ (and hence L_E) via completing it to a zeta integral on two-dimensional adelic spaces and working with the zeta integral using adelic dualities and geometric analytic information, and then apply to the study of

meromorphic continuation and functional equation of $\zeta_{\mathcal{E}}$

the poles of the zeta function: the GRH for $\zeta_{\mathcal{E}}$

the behaviour at the central point: the BSDC for $\zeta_{\mathcal{E}}$

2d adelic analysis can be extended to the general case of regular proper models of smooth projective geometrically irreducible curves of genus g over k. It also gives a new approach to the Arakelov geometry.

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2d class field theory in k-delic form

There are three levels of adelic objects associates to the arithmetic surface $\mathcal{E}$:

$\prod' K_{x,y}$	complete, local-local
$\prod' K_y, \prod' K_x$	incomplete, local-global
K	discrete

Here $\prod'$ is a 2d restricted product,
K_y is the fraction field of the completion of the local ring of $\mathcal{E}$ at a curve y,
K_x is the ring generated by the completion of the local ring of $\mathcal{E}$ at a point x and K,
$K_{x,y}$ is two-dimensional field (product of finitely many fields if x is a singular point of y) associated to $x \in y$;
plus there are some archimedean data involved.

For example, in char 0 the local field associated to a nonsingular point on a fibre (over a prime b of B) is a mixed characteristic 2d local field whose residue field is the local field of positive characteristic associated to the point on the reduction of E modulo the prime b. Two-dimensional local fields are endowed with some special topology which takes into account the topology of the residue field.

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2d local class field theory: for a 2d local field $K_{x,y}$ there is a reciprocity homomorphism

$$K_2(K_{x,y}) \to \text{Gal}(K_{x,y}^{\text{ab}}/K_{x,y})$$

whose kernel $= \cap_{l\geq 1} lK_2(K_{x,y}) =$ intersection of all neighbourhoods of zero in the strongest topology on K_2 which makes its addition continuous and for which the symbol map $K_{x,y}^{\times} \times K_{x,y}^{\times} \to K_2(K_{x,y})$ is sequentially continuous.

It is convenient to work with the topologically separated quotient

$$K_2^t(K_{x,y}) := K_2(K_{x,y})/\cap_{l\geq 1} lK_2(K_{x,y}).$$

The injective reciprocity map $K_2^t(K_{x,y}) \to \text{Gal}(K_{x,y}^{\text{ab}}/K_{x,y})$ can be described explicitly.

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In 2d global class field theory one uses K_2-groups associated to the top and middle level

$$\frac{J_{\mathcal{E}}}{P_{\mathcal{E}}} = \frac{\prod{}' K_2^t(K_{x,y})}{\Delta(\prod{}' K_2(K_y) + \prod{}' K_2(K_x))},$$

where Δ is the map induced by the embeddings $K_x \to K_{x,y}$, $K_y \to K_{x,y}$. In characteristic zero archimedean data are involved as well.

The 2d global reciprocity homomorphism

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Some of the difficulties for 2d adelic theory and the ways to address them

Some of the difficulties from the point of view of 2d adelic analysis:

(1) 2d local fields $K_{x,y}$ are not locally compact spaces, there is no nontrivial real valued translation invariant measure on them,

(2) the structure of $K_2^t(K_{x,y})$ is not known in general,

(3) the data associated to $(x \in y)$ is too ample from the point of view of zeta function purposes which uses every closed point once.

Solutions:

(1) compact and locally compact is not so important, and we can work with $\mathbb{R}((X))$-valued translation invariant measure on $K_{x,y}$ and $K_{x,y}^\times$;

(2) work with $(K_1 \times K_1)(\mathcal{O}_{x,y})$ from which there is a natural in explicit higher class field theory noncanonical surjective homomorphism to $K_2^t(K_{x,y})$;

(3) work with another adelic space associated to all fibres and finitely many horizontal curves on $\mathcal{E}$, with the restricted product taken wrt the integral structure of rank 2.

Some of the difficulties for 2d adelic theory and the ways to address them

Some of the difficulties from the point of view of 2d adelic analysis:

(1) 2d local fields $K_{x,y}$ are not locally compact spaces, there is no nontrivial real valued translation invariant measure on them,

(2) the structure of $K_2^t(K_{x,y})$ is not known in general,

(3) the data associated to $(x \in y)$ is too ample from the point of view of zeta function purposes which uses every closed point once.

Solutions:

(1) compact and locally compact is not so important, and we can work with $\mathbb{R}((X))$-valued translation invariant measure on $K_{x,y}$ and $K_{x,y}^{\times}$;

(2) work with $(K_1 \times K_1)(\mathcal{O}_{x,y})$ from which there is a natural in explicit higher class field theory noncanonical surjective homomorphism to $K_2^t(K_{x,y})$;

(3) work with another adelic space associated to all fibres and finitely many horizontal curves on $\mathcal{E}$, with the restricted product taken wrt the integral structure of rank 2.

Two integral structures in 2d local fields

Let F be a 2d local field whose residue field is a 1d nonarchimedean local field. Denote by $\mathcal{O}$ the ring of integers of F with respect to its discrete valuation of rank 1 and by t_2 a local parameter.

Denote by O the ring of integers with respect to any of its discrete valuations of rank 2. Then O is the preimage of the ring of integers of the residue field. This integral structure O is important for analysis on 2d local fields and for the study of zeta integrals. Denote by t_1 a lift of a local parameter of the residue field. Then $\mathcal{O} = \cup_{j\in\mathbb{Z}} t_1^j O$.

We have the following 2d picture of O-submodules of F:

$$\begin{array}{cccccc}
\cup_j t_2 t_1^j O = t_2\mathcal{O} & \cdots \supset & t_2 t_1^{-1} O \supset & t_2 O \supset & t_2 t_1 O & \supset \cdots \\
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\cup_j t_2^{-1} t_1^j O = t_2^{-1}\mathcal{O} & \cdots \supset & t_2^{-1} t_1^{-1} O \supset & t_2^{-1} O \supset & t_2^{-1} t_1 O \supset \cdots &
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Measure and integration on 2d local fields

Let $\mathcal{A}$ be the ring of sets generated by distinguished sets $a + t_2^i t_1^j O$. Define a function

$$\mu(a + t_2^i t_1^j O) = X^i q^{-j}, \quad q = |O : t_1 O|.$$

Theorem

μ is extended to a well defined finitely additive function on $\mathcal{A}$.

Moreover, for countably many disjoint $A_n \in \mathcal{A}$ such that $\cup A_n \in \mathcal{A}$ and such that $\mu(A_n)$ absolutely converges in $\mathbb{R}((X))$ we get $\mu(A) = \sum \mu(A_n)$.

For a reasonably large class of functions on F which include functions $\sum a_n char_{A_n}$ with distinguished disjoint $A_n \in \mathcal{A}$, such that $\sum a_n \mu(A_n)$ absolutely converges one can define their integral $\int f d\mu$.

This measure and integration theory are compatible with the measure and integration on the residue field.
Extensions of these theory to algebraic groups: Morrow (GL_n), Waller (any alg. group), and a model theoretical work of Hrushovski-Kazhdan in some partial cases.

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Fourier transform on 2d local fields

Self-duality: fix a nontrivial continuous character $\psi\colon F \to \mathbb{C}_1$, then every nontrivial continuous character of F is of the form $x \to \psi(ax)$ for some $a \in F$.

For an integrable function f on F define its Fourier transform

$$\mathcal{F}(f) = \int f(\alpha)\psi(\alpha\beta)d\mu(\alpha).$$

Then

$$\mathcal{F}^2(f)(\alpha) = f(-\alpha).$$

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Two adelic spaces on $\mathcal{E}$

For a curve y on $\mathcal{E}$ and an integer r define an adelic space

$$\mathbb{A}_y^r = \left\{ \sum_{i \ge r} a_i t_y^i : \text{ where } a_i \text{ are nice lifts of } \overline{a_i} \in \mathbb{A}_{k(y)} \text{ to } a_i \in \prod_{x \in y} \mathcal{O}_{x,z} \right\}.$$

Put $\mathrm{A}_y = \cup_r \mathbb{A}_y^r$, $\mathrm{A}_y^0 = \mathbb{A}_y^0$.

Define a large geometric adelic space $\mathrm{A}_{\mathcal{E}}$ associated to the integral structure of rank one on $\mathcal{E}$, it is a subspace of $\prod K_{x,y}$ and is the restricted product of A_y, where y runs through all curves on $\mathcal{E}$, with respect to A_y^0 in the following sense:

$(a_{x,y})_{x \in y \subset \mathcal{E}}$ with $a_{x,y} \in K_{x,y}$ belongs to $\mathrm{A}_{\mathcal{E}}$ if

(a) for almost all y the element $a_{x,y}$ belongs to $\mathcal{O}_{x,y}$ for all $x \in y$ and

(b) there is an integer r such that $(a_{x,y})_{x \in y}$ belongs to $\mathbb{A}_y^r$ for every y.

In characteristic zero some archimedean data involved in the definition of A on horizontal curves.

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Using diagonal embeddings $\prod K_y \to \prod K_{x,y}, \prod K_x \to \prod K_{x,y}$ define $\mathrm{B}_{\mathcal{E}}, \mathrm{C}_{\mathcal{E}}$ as the intersection of $\mathrm{A}_{\mathcal{E}}$ with their images.

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Second type adelic spaces on $\mathcal{E}$

The just defined adelic spaces are quite useful for some algebraic geometric studies. Unfortunately, one cannot integrate on these large adelic spaces.
There are adelic spaces of the second type, on which one can integrate and which are related to the study of the zeta function of the surface.

Put $\mathbb{A}_y = \mathbb{A}_y^0$ and denote $O\mathbb{A}_y = \prod_{x \in y} \mathrm{O}_{x,y} \cap \mathbb{A}_y$, where $\mathrm{O}_{x,y}$ is the preimage in $\prod_{z \in y(x)} O_{x,z}$ with respect to the residue map of the completion of the local ring of y at x; $y(x)$ is the set of local branches of y at x.

Introduce a two-dimensional analytic adelic space $\mathbb{A}_{\mathcal{E}}$ as the restricted product of $\mathbb{A}_y$, $y \in \mathcal{E}$, with respect to the integral structure of rank two $O\mathbb{A}_y$:

an element $(a_{x,y})$, $a_{x,y} \in K_{x,y}$ belongs to $\mathbb{A}_{\mathcal{E}}$ if

(a) for almost all $x \in y$ the element $a_{x,y}$ belongs to $O_{x,y}$;

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In characteristic zero there are also archimedean data involved in the definition of $\mathbb{A}$.

$\mathbb{A}_{\mathcal{E}}$ a subset of $\mathrm{A}_{\mathcal{E}}$, but the adelic structure of the former is not the induced from the latter.

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Second type adelic spaces on $\mathcal{E}$

Fix a subset S' of the set of curves on $\mathcal{E}$ which contains all vertical curves and finitely many horizontal curves. Put

$$\mathbb{A} := \mathbb{A}_{S'} = \mathbb{A}_{\mathcal{E}} \cap \prod_{y \in S'} \mathbb{A}_y.$$

$$\mathsf{A}_{S'} = \mathsf{A}_{\mathcal{E}} \cap \prod_{x \in y \in S'} K_{x,y}.$$

For a curve y put $\mathbb{B}_y = \mathcal{O}_y$ (so its residue field is $k(y)$); for a fibre y define $\mathbb{B}_y$ as the product of the $\mathbb{B}$-spaces for all components of y. Define $\mathbb{B}$ as the intersection of $\prod \mathbb{B}_y$ in $\prod \mathbb{A}_y$ with $\mathbb{A}$.

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Second type adelic spaces on $\mathcal{E}$

We have the following diagramme

$$\begin{array}{ccc} \mathrm{A}_{\mathcal{E}} & - & \mathbb{A}_{S'} \\ | & & | \\ \mathrm{B}_{\mathcal{E}} & - & \mathbb{B}_{S'} \\ | & & \\ K & & \end{array}$$

in which $K = k(S)$.

Duality and adelic measures

For each fibre and curve y choose a complex character $\psi_y = \otimes\psi_{x,y}$ on $\mathbb{A}_y$ trivial on $\mathbb{B}_y$. Put

$$\psi = \otimes\psi_y$$

Choose self-dual measures $\mu_{x,y}$ on $K_{x,y}$ with respect to $\psi_{x,y}$.

Define the measure

$$\mu = \otimes\mu_{x,y}$$

on $\mathbb{A}$.

Define the measure on $\mathbb{B}$ as the product of the measures on $\mathbb{B}_y$ which are the lifts of the discrete measure on $k(y)$.

Define adelic Fourier transform $\mathcal{F}$ using the character ψ and measure μ. We get

$$\int_{\mathbb{B}} f(\alpha\beta)d\mu(\beta) = \frac{1}{|\alpha|}\int_{\mathbb{B}} \mathcal{F}(f)(\alpha^{-1}\beta)d\mu(\beta)$$

Define the measure on $\mathbb{A}^{\times}$ using the local multiplicative measures $(1 - |O_{x,y} : t_1 O_{x,y}|^{-1})^{-1}\mu_{x,y}$.

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Local object T

In the explicit class field theory a major role is played by a surjective homomorphism

$$\mathfrak{t}_{x,y}\colon\ T_{x,y} = (\mathcal{O}_{x,y}\times\mathcal{O}_{x,y})^{\times} \to K_2^t(K_{x,y}) = K_2(K_{x,y})/\cap lK_2(K_{x,y}),$$
$$(t_1^i u, t_1^j v) \mapsto (i+j)\{t_1,t_2\} + \{t_1,u\} + \{v,t_2\}, \qquad u,v\in O_{x,y}^{\times}.$$

Denote by $UK_2^t(K_{x,y})$ the image of $(O_{x,y}\times O_{x,y})^{\times}$. We have a commutative diagramme

$$\begin{array}{ccccc}
 & & \mathcal{O}_{x,y}^{\times}\otimes K_{x,y}^{\times}/\mathcal{O}_{x,y}^{\times} & & \\
 & & \downarrow & \searrow & \\
T_{x,y} & \longrightarrow & \mathcal{O}_{x,y}^{\times}\times\mathcal{O}_{x,y}^{\times}/O_{x,y}^{\times} & \longrightarrow & K_2^t(K_{x,y})/UK_2^t(K_{x,y}).
\end{array}$$

The surjective diagonal map is induced by the symbol map; the vertical map sends $(\alpha, {t_2}^m)$ to $(\alpha^m, 1)$; the composition of the first and second horizontal maps is induced by $\mathfrak{t}_{x,y}$.

Local object T

In the explicit class field theory a major role is played by a surjective homomorphism

$$\mathfrak{t}_{x,y}\colon T_{x,y} = (\mathcal{O}_{x,y} \times \mathcal{O}_{x,y})^{\times} \to K_2^t(K_{x,y}) = K_2(K_{x,y})/\cap lK_2(K_{x,y}),$$
$$(t_1^i u, t_1^j v) \mapsto (i+j)\{t_1, t_2\} + \{t_1, u\} + \{v, t_2\}, \qquad u, v \in O_{x,y}^{\times}.$$

Denote by $UK_2^t(K_{x,y})$ the image of $(O_{x,y} \times O_{x,y})^{\times}$. We have a commutative diagramme

$$\begin{array}{ccccc} & & \mathcal{O}_{x,y}^{\times} \otimes K_{x,y}^{\times}/\mathcal{O}_{x,y}^{\times} & & \\ & & \downarrow & \searrow & \\ T_{x,y} & \longrightarrow & \mathcal{O}_{x,y}^{\times} \times \mathcal{O}_{x,y}^{\times}/O_{x,y}^{\times} & \longrightarrow & K_2^t(K_{x,y})/UK_2^t(K_{x,y}). \end{array}$$

The surjective diagonal map is induced by the symbol map; the vertical map sends $(\alpha, {t_2}^m)$ to $(\alpha^m, 1)$; the composition of the first and second horizontal maps is induced by $\mathfrak{t}_{x,y}$.

Adelic object T

The exact sequence

$$K_2(K_x) \to \oplus K_2(K_{x,z}) \to K_0(k(x)) \to 0,$$

where z runs through branches of curves y at the point x, shows that for the purposes of the unramified class field theory if x is a singular point on a fibre we can use for the global reciprocity map just one copy of $K_2(K_{x,z})$, z a local branch.

Assume from now on that all singular points of fibres are double ordinary. Then the local fields associated to the two local branches at singular points are isomorphic.
Define

$$T = (\mathbb{A} \times \mathbb{A})^\times \cap \prod T_{x,y},$$

where for every point x of a fibre only one copy of $T_{x,z}$ participates, z a local branch. T also includes information which comes from archimedean points on horizontal curves.

We get an adelic homomorphism

$$\mathfrak{t}\colon T \to \prod{}' K_2^t(K_{x,y}).$$

Denote

$$J = \mathfrak{t}(T), \qquad P = P_{\mathcal{E}} \cap J.$$

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We get the following adelic version of the commutative diagramme above

$$\begin{array}{ccccc} & & \mathbb{A}^{\times} \otimes \mathrm{A}_{S'}^{\times}/\mathrm{VA}_{S'}^{\times} & & \\ & & \downarrow & \searrow & \\ T & \longrightarrow & \mathbb{A}^{\times} \times \mathbb{A}^{\times}/V\mathbb{A}^{\times} & \longrightarrow & J/VJ. \end{array}$$

Here

$$\mathrm{VA}^{\times} = \mathrm{A}^{\times} \cap \prod \mathcal{O}_{x,y}^{\times}, \qquad V\mathbb{A}^{\times} = \mathbb{A}^{\times} \cap \prod O_{x,y}^{\times},$$

and VJ is the subgroup built on units.

The diagramme glues together the adelic structures $\mathrm{A}^{\times}$ and $\mathbb{A}^{\times}$.

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Adelic object T_0

Denote $T_0 = T \cap (\mathbb{B} \times \mathbb{B})^\times$.

From the adelic diagramme for T we get a commutative diagramme

$$\begin{array}{ccccc}
 & & \mathbb{B}^\times \otimes \mathsf{B}^\times_{S'}/(\mathsf{B}^\times_{S'} \cap \mathsf{VA}^\times_{S'}) & & \\
 & & \downarrow & \searrow & \\
T_0 & \longrightarrow & \mathbb{B}^\times \times \mathbb{B}^\times/(\mathbb{B}^\times \cap V\mathbb{A}^\times) & \longrightarrow & P/(P \cap VJ),
\end{array}$$

Adelic object T_1

Denote by T_1 the subgroup of T of elements of module 1.
Denote by UT the intersection of T_1 with the product of the nonarchimedean part of $T \cap \prod T_{1x,z}$ and of the archimedean part of T.
The group UT is open in T_1.

The homomorphism $\mathfrak{t}$ induces a surjective map $T/(T_0 + UT) \to J/(P + VJ)$. Note that the unramified theory is described by $J_\mathcal{E}/(P_\mathcal{E} + VJ_\mathcal{E}) \simeq J/(P + VJ)$.

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Zeta integral

For vertical y define $\mathcal{T}_y = T_y = T \cap \prod_{x \in y} T_{x,y}$.

For horizontal y choose a splitting $T_y = T_{1,y} \times N_y$, where $T_{1,y}$ is the kernel of the module map $|\ |$ on T_y. Define $\mathcal{T}_y = T_{1,y} \times N_y^2$.

Define $\mathcal{T} = T \cap \prod \mathcal{T}_y$.

Zeta integral

The general form of 2d zeta (unramified) integral is

$$\zeta(f, |\ |^s) = \int_{\mathcal{T}} f \,||\ ||^s \, d\mu$$

where f is a 2d Bruhat–Schwartz function,
μ is the measure on $K_{x,y}^\times \times K_{x,y}^\times$ or $(\mathbb{A} \times \mathbb{A})^\times$,
$||\ ||$ is the following rescaled module on $\mathcal{T}$: it is the product of $||\ ||_y$ where $||\ ||_y = |\ |_y^{1/2}$ on horizontal y and $||\ ||_y = |\ |_y$ on vertical y.

Example: the local zeta integral on vertical fibres

$$\zeta(char_{(t_1^c O_{x,y}, t_1^{c'} O_{x,y})}, |\ |_2^s, \mu_{x,y}) = q_{x,y}^{d_{x,y}-(c+c')s} \left(\frac{1}{1-q_{x,y}^{-s}}\right)^2$$

where $q_{x,y} = |O_{x,y} : t_1 O_{x,y}|$, and $t_1^{d_{x,y}} O_{x,y}$ is the conductor of $\psi_{x,y}$.

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Zeta integral

If $f = \otimes f_y$ then

$$\zeta(f, |\ |^s) = \prod \zeta_y(f_y, |\ |^s).$$

Assume that every singular point of every fibre is a split ordinary double point and the reduction in residual characteristic 2 and 3 is good or multiplicative.

Then for a centrally normalized function f and vertical fibre $y = \mathcal{E}_b$ over $b \in B$

$$\zeta_y(f, |\ |^s) = |k(b)|^{(f_b + m_b - 1)(1-s)} \zeta_y(s)^2$$

where f_b is the conductor, m_b is the number of irreducible geometric components.

For a horizontal curve y the zeta integral $\zeta_y(f, |\ |^s)$ is a meromorphic function satisfying FE with respect to $s \to 2 - s$, holomorphic outside its poles of multiplicity 2 at $s = 0, 2$, $q^s = 1, q^2$.

In characteristic zero on horizontal curve $\zeta_y(f, |\ |^s) = \hat{\zeta}_{k(y)}(s/2)^2$.

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Zeta integral

Thus, on $\Re(s) > 2$ the zeta integral $\zeta(f, | \ |^s)$ equals the product of $\zeta_{\mathcal{E}}(s)^2$ times an exponential factor which takes into account the conductor of the model $\mathcal{E}$ and times finitely many horizontal zeta integrals. Hence the zeta integral is a holomorphic function on that half plane.

Should we know the functional equation and meromorphic continuation (the RH) for the zeta integral $\zeta(f, | \ |^s)$, it would imply the same properties for the zeta function $\zeta_{\mathcal{E}}(s)$.

$\int_{T_0}$

Denote by $d_y^{-1/2}$ the cardinality of the multiplicative group of the maximal finite subfield of $k(y)$.

For a function g on $\mathbb{A} \times \mathbb{A}$ such that for almost every fibre y the integral

$$d_y \int g \, d\mu_{(\mathbb{B}_y \times \mathbb{B}_y)^\times} = 1$$

(for example $g = \otimes char_{O_{x,y} \times O_{x,y}}$), define

$$\int_{T_0} g \, d\mu_{T_0} = \lim d_{S_o} \prod_{y \in S_o} \int g \, d\mu_{(\mathbb{B}_y \times \mathbb{B}_y)^\times}$$

where d_{S_o} is the product of all d_y attached to vertical fibres in finite $S_o \subset S'$.

So the measure on T_0 is the tensor product of the rescaled fibre measures, and is not the lift of the discrete measure.

$\int_{\partial T_0}$

The weak boundary of $T_{0,y} = (\mathbb{B}_y \times \mathbb{B}_y)^\times$ is $\mathbb{B}_y \times \mathbb{B}_y \setminus (\mathbb{B}_y \times \mathbb{B}_y)^\times$.

Let ∂T_0 be the weak boundary of T_0, i.e. the union of the product of the weak boundaries of $T_{0,y}$ with y in a finite subset of fibres and horizontal curves and the product of $T_{0,y}$ at all other y.

Define the integral $\int_{\partial T_0}$ similarly to the definition of $\int_{T_0}$.

2d theta formula

Theorem

For a centrally normalized f its transform can be written

$$\mathcal{F}(f)(\alpha) = f(\nu^{-1}\alpha), \qquad |\nu| = 1.$$

We get

$$\int_{T_0} \big(f(\alpha\beta) - |\alpha|^{-1} f(\nu^{-1}\alpha^{-1}\beta) \big) \, d\mu_{T_0}(\beta)$$
$$= \int_{\partial T_0} \big(|\alpha|^{-1} f(\nu^{-1}\alpha^{-1}\beta) - f(\alpha\beta) \big) \, d\mu_{\partial T_0}(\beta).$$

This formula glues together differently normalized structures on vertical and horizontal curves.

2nd calculation of the zeta integral

Put $N = |T|$.

Split N into the disjoint union of metrizable spaces N^+ and N^-, such that the involution $x \to x^{-1}$ maps one of them onto the other.

Denote by $\mathcal{T}_1$ the kernel of $|| \ ||$ on $\mathcal{T}$.

In the proof of 2d T–I formula we will use the filtration $\mathcal{T} > \mathcal{T}_1 > T_0$ and 2d theta formula.

For the zeta integral we have

$$\zeta(f, |\ |^s) = \int_N \zeta_n(f, |\ |^s)\, d\mu_N(n)$$

where

$$\zeta_n(f, |\ |^s) = n^s \int_{\mathcal{T}_1} f(m_n\alpha)\, d\mu(\alpha)$$

and $\{m_n\}_{n\in N}$ is a subgroup of $\mathcal{T}_{y_0}$, y_0 is the image of the zero section, such that $||m_n|| = n$.

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2d IT

Theorem

(2d version of the Tate–Iwasawa unramified theory)
On the half plane $\Re(s) > 2$ the zeta integral is the sum of three terms

$$\zeta(f, |\ |_2^s) = \xi(|\ |_2^s) + \xi(|\ |_2^{2-s}) + \omega(|\ |_2^s).$$

The term $\xi((|\ |_2^s) = \int_{N^+} \zeta_n(f, |\ |_2^s)\, d\mu_{N^+}(n)$ is an entire function on the complex plane.
The boundary term $\omega(|\ |_2^s) = \int_{N^-} \omega_n(|\ |_2^s)\, d\mu_{N^-}(n)$ is given by

$$\begin{aligned}
\omega_n(|\ |_2^s) &= n^s \int_{\mathcal{T}_1} \big(f(m_n\alpha) - n^{-2} f(m_n^{-1}\alpha^{-1})\big)\, d\mu(\alpha) \\
&= n^s \int_{\mathcal{T}_1/T_0} \int_{T_0} \big(f(m_n\gamma\beta) - n^{-2} f(m_n^{-1}\gamma^{-1}\beta)\big)\, d\mu_{T_0}(\beta)\, d\mu_{\mathcal{T}_1/T_0}(\gamma) \\
&= n^{s-2} \int_{\mathcal{T}_1} \big(|\alpha|^{-1} - 1\big)\, f(m_n^{-1}\alpha^{-1})\, d\mu(\alpha) \\
&+ n^s \int_{\mathcal{T}_1/T_0} \int_{\partial T_0} \big(n^{-2}|\gamma|^{-1} f(m_n^{-1}\nu^{-1}\gamma^{-1}\beta) - f(m_n\gamma\beta)\big)\, d\mu_{\partial T_0}(\beta)\, d\mu_{\mathcal{T}_1/T_0}(\gamma).
\end{aligned}$$

Integral representation of the boundary term

Write

$$\omega(|\ |_2^s) = \int_{N^-} h(n)\, n^{s-2}\, d\mu_{N^-}(n),$$

where

$$h(n) = \int_{\mathcal{T}_1/T_0} \left(\int_{\partial T_0} \left(n^2 f(m_n \gamma \beta) - f(m_n^{-1} \nu^{-1} \gamma^{-1} \beta)\right) d\mu_{\partial T_0}(\beta) \right) d\mu_{\mathcal{T}_1/T_0}(\gamma).$$

Hence $\omega(|\ |_2^s)$ is the *Laplace–Stieltjes transform* $\int_0^\infty e^{-st}\, dj(t)$ of an appropriate function $j(t)$ which is a modification of h.

FE for the function h:

$$h(n^{-1}) = -n^{-2} h(n).$$

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Integral representation of the boundary term

For example, for an elliptic curve E be over $\mathbb{Q}$ write

$$c_{\mathcal{E}}^{1-s}\,\zeta_{\mathcal{E}}(s)^2 = \sum_{n\in c_{\mathcal{E}}\mathbb{N}} \frac{d(n^2)}{n^s}.$$

Then

$$h(x) = -\sum_{n\in c_{\mathcal{E}}\mathbb{N}} d(n^2)V_{n^2}(x)$$

where

$$V_m(x) = 4\sum_{l\geq 1}\sigma_0(l)\big(K_0(2\pi lmx^{-2}) - x^2K_0(2\pi lmx^2)\big)$$

and

$$K_0(x) = \frac{1}{2}\int_0^\infty e^{-x(t+\frac{1}{t})/2}\,\frac{dt}{t}.$$

Applications: 1. Meromorphic continuation and FE

The (conjectural in general) automorphic property of the L-function of E is lost when one works with the zeta function.
Which analytic shape should take the function h so that its transform has meromorphic continuation and FE?

Let X be a space of complex valued functions on the real line in which the Hahn-Banach theorem holds.

Definition

A function $g \in X$ is called X-*mean-periodic* if it satisfies one of the equivalent conditions:

there exists a closed proper linear subspace of X which contains all translates of g;

g is a solution of a homogeneous convolution equation $g * \tau = 0$ where τ is a non-zero element in the dual space of X.

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For example, X can be the space of continuous functions $\mathcal{C}(\mathbb{R})$, the space of smooth functions $\mathcal{C}^\infty(\mathbb{R})$, the space $\mathcal{C}^\infty_{\text{exp}}(\mathbb{R})$ of smooth functions of exponential growth. Similarly one can define mean-periodicity in spaces like the space $\mathcal{F}_{\text{exp}}(\mathbb{Z})$ of functions of exponential growth on integers.

In all these spaces the property of harmonic synthesis holds: every mean-periodic function g is approximated by exponential polynomials belonging to the subspace generated by translations of g.

The theory of mean-periodic functions was developed in the second half of the 20th century and so it is 100 years younger than the theory of modular functions.
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Let g be mean-periodic in one of those X, $g * \tau = 0$, and let g be of finite exponential growth. Define

$$g^+(t) = g(t) \text{ for } t > 0, \quad g^+(0) = g(0)/2, \quad g^+(t) = 0 \text{ for } t < 0.$$

Then $g^+ * \tau \in X^*$ and for sufficiently large $\Re(s)$ the Laplace–Stieltjes transform of g equals

$$G(s) = \frac{\int_{-\infty}^{\infty} g^+ * \tau(t)\, e^{-st}\, dt}{\int_{-\infty}^{\infty} \tau(t)\, e^{-st}\, dt}.$$

This does not depend on the choice of $\tau \neq 0$. Both the numerator and denominator extend to entire functions on the plane, and hence $G(s)$ has meromorphic extensions to the plane.
It is called the *Laplace–Stieltjes–Carleman transform* of g.

If the original mean-periodic function is odd then its L-C transform is an even function. So if $h(exp(-t))$ is mean-periodic then the boundary term extends to a meromorphic function satisfying the functional equation wrt $s \to 2 - s$.

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Applications: 1. Meromorphic continuation and FE

In a recent work with G. Ricotta and M. Suzuki it is shown that if the zeta function $\zeta_X(s)$ of a regular scheme X of dimension n extends to a meromorphic function on the complex plane with the expected (conjectural in general) analytic shape, and satisfies a functional equation with sign ϵ, then there exists an integer $m_X \geq 1$ such that for every integer $m \geq m_X$ the function $h_{X,m}(\exp(-t)))$ is mean-periodic in $\mathcal{C}^{\infty}_{\exp}(\mathbb{R})$. Here

$$h_{X,m,\epsilon}(x) = f_{X,m}(x) - \epsilon x^{-1} f_{X,m}(x^{-1})$$

where $f_{X,m}(x)$ is the inverse Mellin transform of $\hat{\zeta}_X(s/n)\hat{\zeta}_{\mathbb{Q}}(s)^m$, where hat stands for the appropriately completed zeta functions.

Conversely, if the function $h_{X,m_X,\epsilon}(\exp(-t)))$ is mean-periodic then $\zeta_X(s)$ has a meromorphic continuation and satisfies the expected functional equation with sign ϵ.

Applications: 1. Meromorphic continuation and FE

In particular, as a consequence we get a correspondence

$$\mathcal{C}: \quad X \quad \mapsto \quad h_{X,m_X,\epsilon}(\exp(-t))$$

from the set of arithmetic schemes whose Hasse zeta function $\zeta_X(s)$ has the expected analytic properties to the space of mean-periodic functions in $\mathcal{C}^\infty_{\exp}(\mathbb{R})$.

This correspondence between mean-periodic functions in $\mathcal{C}^\infty_{\exp}(\mathbb{R})$ and the class of certain Dirichlet series which admit a meromorphic continuation and functional equation and which contains arithmetic zeta functions and their quotients extends the Hecke–Weil correspondence between modular forms and L-functions.

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Applications: 1. Meromorphic continuation and FE of $\zeta_{\mathcal{E}}$

Now we state a hypothesis which naturally comes in the study of the zeta integrals in 2d adelic analysis.

Hypothesis

The function

$$H(t) = \begin{cases} h(e^{-t}), & \text{in characteristic zero,} \\ h(q^{-t}), t \in \mathbb{Z}, & \text{in positive characteristic,} \end{cases}$$

is a mean-periodic function in the space $\mathcal{C}^{\infty}_{\exp}(\mathbb{R})$ if K is of characteristic zero and in the space $\mathcal{F}_{\exp}(\mathbb{Z})$ if K is of positive characteristic.

Applications: 1. Meromorphic continuation and FE of $\zeta_{\mathcal{E}}$

Theorem

If the Hypothesis is true then the boundary term and the zeta integral and hence the zeta and L-functions have meromorphic continuation and satisfy the functional equation wrt $s \to 2 - s$.

In the positive characteristic mean-periodicity of $H(t)$ easily follows from the known rationality and meromorphic continuation of the zeta function.

In characteristic zero mean-periodicity of $H(t)$ for elliptic curves over totally real fields follows from their potential automorphic property

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Applications: 2. GRH for the zeta integral

In this section assume that there is just one horizontal curve – the zero section – in the set of curves S' in the definition of the zeta integral. Then the zeta integral has (first from the right) pole of order 4 at $s = 2$.

Recall that in characteristic 0

$$H(t) = \int_{\mathcal{T}_1/T_0} \left(\int_{\partial T_0} \left(e^{-2t} f(e^{-t}\gamma\beta) - f(e^{t}\nu^{-1}\gamma^{-1}\beta) \right) d\mu_{\partial T_0}(\beta) \right) d\mu_{\mathcal{T}_1/T_0}(\gamma).$$

This involves an integral over the boundary ∂T_0 which is very large, unlike d1 case. It is natural to expect a smoothening effect for individual oscilations of the integrand function which can be expressed in monotone properties of H and its derivatives.

It is easy to check that $H(t)$ and its first derivatives $H'(t)$, $H''(t)$, $H'''(t)$ keep their sign for all sufficiently large t.

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In the study of the zeta integral the following hypothesis comes naturally.

Hypothesis

The fourth derivative of H keeps its sign near infinity.

Theorem

If the hypothesis holds and if the zeta function does not have real poles in the strip $\Re(s) \in (1, 2)$ then the zeta function does not have complex poles in the same strip.

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For computational data see http://www.maths.nott.ac.uk/personal/ibf/tbl.html

To check the absence of real poles on $(1, 2)$ for an individual elliptic curve is very easy computationally and it is known for all elliptic curves over rationals of conductor smaller 20000.

We also have

Theorem

M. Suzuki: Assume that E is an elliptic curve over an algebraic number field and its L-function satisfies a meromorphic continuation and functional equation, the GRH holds for the L-function, all nonreal zeros of L on the critical line are of multiplicity not greater than $1+$ the multiplicity of the real zero of L at $s = 1$. Also assume some technical estimate on the derivative of L holds. Then $H''''(t)$ is positive near infinity.

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Applications: 3. BSDC

To compute the local behaviour of $\zeta_{\mathcal{E}}(s)$ at $s = 1$ we assume that the zeta function has a meromorphic continuation and FE.

2d adelic analysis reduces the study of the pole of the zeta integral at $s = 1$ to the study of the pole of the boundary term at $s = 1$.
The latter involves an integral over the boundary ∂T_0 of a certain function related to f.

Thus, information about the boundary ∂T_0 allows one to compute the order of the pole of the zeta integral (and hence the zeta function) at $s = 1$.

The space T_0 modulo units can be studied using the commutative diagramme for T_0 and the top object $\mathbb{B}^\times \otimes \mathrm{B}_{S'}/(\mathrm{B}_{S'} \cap \mathrm{VA}_{S'}^\times)$ in it.

The quotient of $\mathrm{B}_{S'}/(\mathrm{B}_{S'} \cap \mathrm{VA}_{S'}^\times)$ by the image of $K^\times$ and by $p^*\mathrm{Pic}(B)$, where $p\colon \mathcal{E} \to B$, is a finitely generated group (in positive characteristic the Neron–Severi group of $\mathcal{E}$ modulo its subgroup generated by one nonsingular fibre).

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Through the boundary term and the two adelic objects structures one directly relates the analytic and arithmetic ranks of E.

See section 58 of Analysis on arithmetic schemes.II and Analysis on arithmetic schemes.III for more detail.

In positive characteristic things are already understood quite well.

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Applications: 4. Automorphic forms on surfaces

See the last section of Adelic approach to the zeta function of arithmetic schemes in dimension two

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See https://www.maths.nott.ac.uk/personal/ibf/mp.html for pdf files of papers

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