Introducing anabelian geometry

Ivan Fesenko

1D anabelian geometry

Theorem (Neukirch, Iwasawa, Ikeda, Uchida). If K_1, K_2 are number fields then

$$\text{Ring-Iso}(K_1, K_2) \simeq \text{TopGroup-Iso}(G_{K_1}, G_{K_2})/\text{Inn}(G_{K_2}).$$

Neukirch's CFT mechanism was influenced by his previous work in anabelian geometry.

Note that two non-isomorphic finite extensions of $\mathbb{Q}_p$ can have isomorphic absolute Galois groups:

Theorem (Jarden–Ritter) Two absolute Galois groups of local number fields F_1 and F_2 with odd residue characteristic are topologically isomorphic iff their degrees over $\mathbb{Q}_p$ are the same and the degree of the maximal abelian extension of $\mathbb{Q}_p$ in each of them are the same.

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Including ramification filtration information helps:

Theorem (Mochizuki). If F_1, F_2 are local fields then

$$\text{Ring-Iso}(F_1, F_2) \simeq \text{FiltrTopGroup-Iso}(G_{F_1}, G_{F_2})/\text{Inn}(G_{F_2}).$$

2D anabelian geometry

Anabelian geometry for hyperbolic curves over number fields was proposed by Grothendieck and pioneered by Nakamura, Tamagawa, Mochizuki.

Anabelian geometry includes

bi-anabelian geometry (restoring isomorphism classes of scheme objects from isomorphisms of their fundamental groups)

relative anabelian geometry (restoring k-scheme objects or their isomorphism classes from their fundamental group mapped to the fundamental group of k)

absolute anabelian geometry (restoring schemes or their isomorphism classes from fundamental groups via topological group algorithms)

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2D anabelian geometry

Tamagawa:

> *mono-anabelian geometry involves a single object while bi-anabelian geometry involves two objects.*
>
> *But, in fact, mono-anabelian geometry should be considered as involving all objects...*
>
> *to say an algorthim is purely group-theoretic, it must universally and uniformly apply to all profinite groups isomorphic to Galois/fundamental groups of schemes in a specified category, forgetting any isomorphism between the profinite group in question and the Galois/fundamental group.*
>
> *In this sense, mono-anabelian geometry should be regarded as "pan-anabelian geometry"*

Algebraic fundamental groups

For any geometrically integral (quasi-compact) scheme X over a perfect field k we have the following fundamental exact sequence

$$1 \to \pi_1^{\mathrm{geom}}(X) \to \pi_1(X) \to \pi_1(\mathrm{Spec}(k)) = G_k \to 1.$$

Where $\pi_1(X)$ is the *algebraic fundamental group* of X,
$\pi_1^{\mathrm{geom}}(X) = \pi_1(X \times_k k^{\mathrm{alg}})$,
k^{alg} is an algebraic closure of k.

Suppressed dependence of the fundamental groups on basepoints actually means that objects are often well-defined only up to conjugation by elements of $\pi_1(X)$.

Algebraic fundamental groups of schemes over number fields (or fields closely related to number fields, such as local fields or finite fields) are also called *arithmetic fundamental groups*.

Examples.

$\pi_1(\mathbb{P}^1) = \pi_1(\mathbb{P}^1 \setminus \{0\}) = 1$,
$\pi_1(\mathbb{P}^1 \setminus \{0,1\}) = \hat{\mathbb{Z}}$,
$\pi_1(E) = \hat{\mathbb{Z}} \times \hat{\mathbb{Z}}$, E an elliptic curve.

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Hyperbolic curves

If C is a complex irreducible smooth projective curve minus a finite set of its points, then $\pi_1(C)$ is isomorphic to the profinite completion of the topological fundamental group of the Riemann surface associated to C.

Recall that a *hyperbolic curve* C over a field k of characteristic zero is a smooth projective geometrically connected curve of genus g minus r points such that the Euler characteristic $2-2g-r$ is negative.

Examples include a projective line minus three points or an elliptic curve minus one point.

The algebraic fundamental group of a hyperbolic curve is nonabelian.

If C is the result of base-changing a curve over a field k to the field of complex numbers, then the fundamental sequence for such a curve over k induces a homomorphism ψ from G_k to the quotient group $\mathrm{Out}(\pi_1^{\mathrm{geom}}(C))$ of the automorphism group of $\pi_1^{\mathrm{geom}}(C)$ by its normal subgroup of inner automorphisms.

Theorem (Belyi (and Bogomolov)). An irreducible smooth projective algebraic curve C defined over a field of characteristic zero can be defined over an algebraic closure $\mathbb{Q}^{\mathrm{alg}}$ if and only if there is a covering $C \longrightarrow \mathbb{P}^1$ which ramifies over no more than three points of $\mathbb{P}^1$.

This theorem implies that ψ is an embedding of $G_{\mathbb{Q}}$ into the Out group of the pro-finite completion of a free group with two generators.

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Rigidity in anabelian geometry

Anabelian geometry "yoga" for so-called anabelian schemes of finite type over a ground field K (such as a number field, a field finitely generated over its prime subfield, etc.) states that

an anabelian K-scheme X can be recovered from the topological group $\pi_1(X)$ and the surjective homomorphism of topological groups $\pi_1(X) \longrightarrow G_K$ (up to purely inseparable covers and Frobenius twists in positive characteristic).

Thus, the algebraic fundamental groups of anabelian schemes are *rigid*.

This can be compared with

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Two questions raised by Grothendieck

Grothendieck proposed the following questions:

Q1. Are hyperbolic curves over number fields or finitely generated fields anabelian?

A point x in $X(k)$, i.e. a morphism $\mathrm{Spec}(k) \longrightarrow X$, determines, in a functorial way, a continuous section $G_k \longrightarrow \pi_1(X)$ (well-defined up to composition with an inner automorphism) of the surjective map $\pi_1(X) \longrightarrow G_k$.

Q2. The section conjecture asks if, for a geometrically connected smooth projective curve X over K, of genus > 1, the map from rational points $X(K)$ to the set of conjugacy classes of sections is surjective (injectivity was already known).

2D anabelian geometry

The Neukirch–Ikeda–Uchida theorem is a birational version of Q1 in the lowest dimension.

A similar birational recovery property for fields finitely generated over $\mathbb{Q}$ was proved by Pop.

Later Mochizuki proved a similar recovery property for a subfield of a field finitely generated over $\mathbb{Q}_p$.

With respect to Q1, important contributions were made by Nakamura and Tamagawa.

Then Mochizuki proved that hyperbolic curves over finitely generated fields of characteristic zero are indeed anabelian.

Using nonarchimedean Hodge–Tate theory, Mochizuki proved that a hyperbolic curve X over a subfield K of a field finitely generated over $\mathbb{Q}_p$ can be recovered functorially from the canonical projection $\pi_1(X) \longrightarrow G_K$.

The section conjecture in Q2 has not been established.

A combinatorial version of the section conjecture is established by Hoshi and Mochizuki.

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Key theorems in 2D anabelian geometry

Let C be a hyperbolic curve over k.

For a point x of $\overline{C}(k^{\rm alg}) \setminus C(k^{\rm alg})$ the inertia group I_x is the stabiliser group of an extension of x to the maximal unramified extension of $k(C) \otimes k^{\rm alg}$, a subgroup of $\pi_1^{\rm geom}(C)$.

The group I_x is pro-cyclic.

The normaliser of I_x in $\pi_1(C)$ is the decomposition group D_x of x.

Theorem (Namakura). If K is a number field then every pro-cyclic subgroup of $\pi_1^{\rm geom}(C)$ is a subgroup of some I_x, $x \in \overline{C}(K^{\rm alg}) \setminus C(K^{\rm alg})$.

Theorem (Tamagawa, Mochizuki). Let K be a number field or more generally a subfield of a finitely generated extension of $\mathbb{Q}_p$.

Then for two hyperbolic curves X, Y over K the map from K-isomorphisms $X \longrightarrow Y$ to continuous open G_K-isomorphisms of profinite groups $\pi_1(X) \to \pi_1(Y)$ modulo inner conjugation by $\pi_1(Y)$ is a bijection.

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An absolute version of the previous theorem

An absolute version of this theorem (following previous work, the strongest versions were established by Mochizuki):

If K is a finitely generated field over $\mathbb{Q}$ then for two hyperbolic curves X, Y over K the map from isomorphisms $X \longrightarrow Y$ to continuous open isomorphisms of profinite groups $\pi_1(X) \longrightarrow \pi_1(Y)$ modulo inner conjugation by $\pi_1(Y)$, is a bijection.

Powerful reconstruction

Definition. Two hyperbolic curves C_1, C_2 over fields k_1 and k_2 are called isogenous if there is a hyperbolic curve C over a field k with finite étale morphisms $C \longrightarrow C_i$.

Theorem (Mochizuki).
For a hyperbolic curve C over a number field k isogenous to a hyperbolic curve of genus zero (e.g. an elliptic curve with one point removed) there is a universal functorial group theoretical algorithm to reconstruct the field l from the topological group $\pi_1(C_l)$, where $C_l = C \times_k l$, $l = k$ or any of its non-archimedean completions.

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Powerful reconstruction

The proof reconstructs

- $\pi_1^{\mathrm{geom}}(C)$
- $K^\times$
- $K_0^\times$
- non-archimedean valuations of K_0
- K
- $K_0^{\mathrm{alg}}(C_0)$
- $K(C)$

using

- Belyi cuspidalisation
- Nakamura's results
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Powerful reconstruction

And it uses the following two topological group theoretical properties of the absolute Galois group of a number field or of its nonarchimedean completion and of arithmetic fundamental groups:

$\diamond$ each of its open subgroups is centre-free (slimness),

$\diamond$ each nontrivial normal closed subgroup H of any open subgroup, with the property that H is topologically finitely generated as a group, is open (elasticity).

On cyclotomes in mono-anabelian geometry

Let F be a finite extension of $\mathbb{Q}_p$.

Let G be a topological group isomorphic to G_F.

Theorem. There is a group theoretical algorithm (for example, using local CFT) to produce from the group G a G-monoid $O^{\triangleright}(G)$ isomorphic to the G_F-monoid $O^{\triangleright}$.

Denote by $\Lambda(M)$ the projective limit of n-torsion elements of M, $n \geqslant 1$.

The local reciprocity map induces the cyclotomic rigidity isomorphism $\Lambda(O^{\triangleright}) \simeq \Lambda(G_F)$.

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On cyclotomes in mono-anabelian geometry

Theorem (Mochizuki). Let G be a topological group isomorphic to G_F.

Let M be a G-monoid isomorphic to the G_F-monoid $O^{\triangleright}$.

Then $\mathrm{Aut}(G \curvearrowright M) \simeq \mathrm{Aut}(G)$.

There is a functorial algorithm producing from the G-monoid M a G_F-isomorphism $\Lambda(M) \simeq \Lambda(O^{\triangleright}) \simeq \Lambda(G_F)$, using the cyclotomic rigidity isomorphism.

Moreover, this is the unique isomorphism $\Lambda(M) \simeq \Lambda(G_F)$ such that the following diagramme is commutative:

$$\begin{tikzcd} M \arrow[r, hook] \arrow[d] & \varinjlim_{H \leqslant G_F} H^1(H, \Lambda(M)) \arrow[d] \\ O^{\triangleright}(G) \arrow[r, hook] & \varinjlim_{H \leqslant G_F} H^1(H, \Lambda(G_F)) \end{tikzcd}$$

where horizontal arrows are given by Kummer theory.

Applications of anabelian geometry

The powerful restoration results in absolute mono-anabelian geometry of certain hyperbolic curves over number fields and local fields are applied in the IUT theory.

See slides on IUT workshops in 2015, 2016,

and slides and videos of the 4 recent RIMS workshops on anabelian geometry and IUT:

`https://www.kurims.kyoto-u.ac.jp/~motizuki/project-2021-english.html`.