Higher adelic theory

Ivan Fesenko

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CFT and its three main generalisations

CFT = Class Field Theory, HCFT = Higher CFT, HAT = Higher Adelic Theory,

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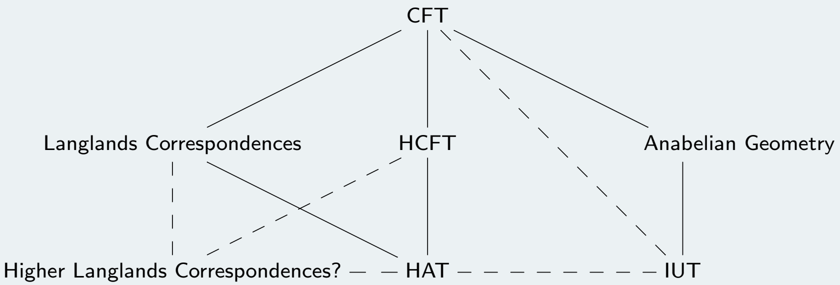

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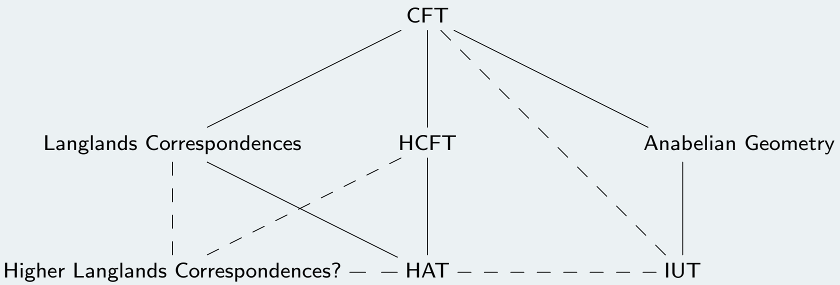

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Back to the root: CFT

$$\begin{tikzcd}[column sep=small] & \text{CFT} \arrow[dl, no head] \arrow[dr, no head] & \\ \text{SCFT} & & \text{GCFT} \end{tikzcd}$$

SCFT = special CFT

Using torsion/division points or values of appropriate functions at torsion points to generate finite extensions of the base fields under investigation and to describe the Galois action on them.

Cyclotomic: Kronecker, Weber, Hilbert.

Using elliptic curves with CM: Kronecker, Weber, a relevant portion of Takagi's work, ...

Using abelian varieties with CM: Shimura.

These theories are not extendable to arbitrary number fields. They are not functorial.

Hilbert Problem 12 was about extensions of SCFT to number fields, the best was achieved by Shimura.

Local SCFT using Lubin–Tate formal groups works over any local field with finite residue field and does not work over local fields with infinite perfect residue field.

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GCFT = general CFT

These theories follow very different conceptual patterns than SCFT.

The list of GCFTs for arithmetic fields includes:

Takagi 1920, the first work in GCFT with his general existence theorem and its applications;

Artin reciprocity map, whose full construction uses Chebotarev's theorem;

Hasse, the use of the Brauer group in CFT, the first local CFT, local-to-global aspects;

Chevalley's invention of idèles, local-to-global, the global reciprocity map as the product of the local reciprocity maps, whose kernel contains the diagonal image of global elements.

Classical approaches to CFT are presented, among many sources, in Hasse's Klassenkörperbericht, and in Weil's and Lang's books.

Cohomological approaches: Artin–Tate, ...

Finding explicit formulas for the Hilbert pairing and its generalisations (Hilbert Problem 9) was one of the ways to get more explicit information about the reciprocity map and to apply CFT.

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Tate–Dwork, Hazewinkel, Neukirch, F

These theories:

◇ clarified and made explicit some of the key structures of CFT

◇ they are less dependent on torsion and they do not use the Brauer group

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◇ they really explain CFT.

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CFT mechanism

CFT mechanism discovered by Neukirch.

Start with an abelian topological group A endowed with a continuous action by a profinite group G.

Think of G as the absolute Galois group G_k of a field k.

Assume (a) that there is a surjective homomorphism

$$\deg\colon G_k \to \hat{\mathbb{Z}}.$$

Denote its kernel by $G_{\tilde{k}}$.

Then for an open subgroup G_K of G_k we get a surjective homomorphism

$$\deg_K = |G_k : G_K G_{\tilde{k}}|^{-1} \deg\colon G_K \to \hat{\mathbb{Z}}.$$

Any element of G_K which is sent by $\deg_K$ to $1 \in \hat{\mathbb{Z}}$ is called a frobenius element w.r.t. $\deg_K$.

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Assume (b) that there is a homomorphism

$$v\colon A_k \to \hat{\mathbb{Z}}, \qquad v(A_k) = \mathbb{Z} \quad \text{or} \quad v(A_k) = \hat{\mathbb{Z}}$$

such that for open subgroups G_K of G_k

$$v(N_{K/k} A^{G_K}) = |G_k : G_K G_{\tilde{k}}|\, v(A_k).$$

Denote

$$A_K := A^{G_K}, \quad v_K := |G_k : G_K G_{\tilde{k}}|^{-1} v \circ N_{K/k}\colon A_K \to \hat{\mathbb{Z}}.$$

Extensions of K inside $K\tilde{k}$ can be viewed as 'unramified' extensions wrt $(\deg, v)$.

Call $\pi_K \in A_K$ such that $v_K(\pi_K) = 1$ a prime element of A_K.

For a finite extension K of k and a finite Galois extension L/K and σ in its Galois group choose any $\tilde{\sigma} \in G(L\tilde{k}/K)$ such that

$$\deg(\tilde{\sigma}) \in \mathbb{N}_{\geq 1} \qquad \text{and } \tilde{\sigma}|_L = \sigma.$$

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CFT mechanism

The pair $(\deg, v)$ defines an *explicit reciprocity map* for $G_L \trianglelefteq G_K \le G_k$, $G(L/K) = G_K/G_L$,

$$\Psi_{L/K} \colon G(L/K) \to A^{G_K}/N_{L/K}A^{G_L}, \quad \sigma \mapsto N_{\Sigma/K}\pi_\Sigma \mod N_{L/K}A_L$$

where Σ is the fixed field of $\tilde{\sigma}$ and π_Σ is a prime element of A_Σ.

If appropriate axioms for A under the action of G (axioms of CFT) are satisfied, then

◇ $\Psi_{L/K}$ is well defined, and it induces an isomorphism $G(L/K)^{\mathrm{ab}} \to A_K/N_{L/K}A_L$,

◇ $\Psi_{L/K}$ satisfies all standard functorial properties of CFT.

This CFT mechanism is purely group theoretical and does not depend on ring structures. However, to verify the CFT axioms for concrete fields one has to use ring structures.

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CFT mechanism

In CFT of local fields with finite residue field one takes the maximal unramified extension of $\mathbb{Q}_p$ or the maximal constant extension as $\tilde{k}/k$; $\Psi_K \colon K^\times \to G_K^{\mathrm{ab}}$.

In CFT of global fields one takes the only $\hat{\mathbb{Z}}$-subextension of the maximal abelian extension of $\mathbb{Q}$ or the maximal constant extension as $\tilde{k}/k$; $\Psi_K \colon \mathbb{A}_K^\times/K^\times \to G_K^{\mathrm{ab}}$.

Classical study of class formations aimed to derive CFT from as few axioms as possible.

The long term search for class formations can be interpreted as distinguishing purely monoid theoretical aspects of CFT (CFT mechanism) from its ring theoretical aspects (proving axioms of CFT).

Remark. In his explicit GCFT Neukirch was partially motivated by his work in anabelian geometry of number fields.

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Anabelian geometry and IUT

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1D theory for global fields (Neukirch, Iwasawa, Ikeda, Uchida),

Neukirch's CFT mechanism was influenced by his previous work in anabelian geometry.

Anabelian geometry for hyperbolic curves over number fields was proposed by Grothendieck and pioneered by Nakamura, Tamagawa, Mochizuki.

Anabelian geometry includes

bi-anabelian geometry (restoring isomorphism classes of scheme theoretic objects) and

mono-anabelian geometry (restoring scheme theoretic objects).

These theories are group theoretical, algorithmic and explicit, features similarly to CFT mechanism.

Powerful restoration results in absolute mono-anabelian geometry were established by Mochizuki and applied in the IUT theory.

Watch Porowski's talk for basic anabelian geometry.

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'Pre-Takagi' LC

Currently, the main arithmetic achievements in arithmetic LC are of *special type only*.

100 years after Takagi's pioneering work that started GCFT and 50 years after the beginning of LC we are still awaiting for results of general type in arithmetic LC.

Despite some partial success, most fundamental problems in arithmetic LC remain open.

In particular,
Shimura–Taniyama conjecture over arbitrary number fields is open,
functoriality is open,
purely local presentation of the local LC, even for $GL(n)$ for all n, is open,
the full $GL_2(\mathbb{Q})$ case is open.

L. Lafforgue proved the equivalence between functoriality in LC and the existence of a certain *non-additive* Fourier transforms satisfying a Poisson formula.

This reformulation asks for a definition of the Fourier transform on functional spaces for a general reductive algebraic group where one cannot use the obvious relation of the general linear group to matrix ring.

This group theoretical aspect in the absence of ambient ring structure reminds some aspects of anabelian geometry.

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Linearity of LC

One can view LC as a *linear* theory over abelian CFT. Since it is a representation theory, LC inevitably misses various important features of the full absolute Galois group that are not of linear representation type.

For example, anabelian geometry uses the following two group theoretical properties of the absolute Galois group of a number field or of its nonarchimedean completion: each of its open subgroups is centre-free, each nontrivial normal closed subgroup H of any open subgroup, with the property that H is topologically finitely generated as a group, is open. These properties are not used in LC.

Question. Can the use of non-linear theories, HCFT and anabelian geometry, help with new understanding of LC, including its expected development of general type?

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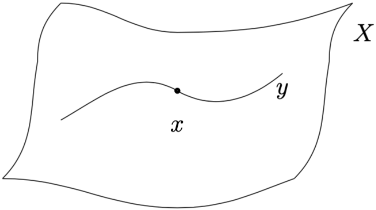
For example, anabelian geometry uses the following two group theoretical properties of the absolute Galois group of a number field or of its nonarchimedean completion: each of its open subgroups is centre-free, each nontrivial normal closed subgroup H of any open subgroup, with the property that H is topologically finitely generated as a group, is open. These properties are not used in LC.

Question. Can the use of non-linear theories, HCFT and anabelian geometry, help with new understanding of LC, including its expected development of general type?

See `https://ivanfesenko.org/wp-content/uploads/2021/11/232.pdf` for various problems related to unification of the generalisations of CFT.

$x \in y \subset X$



2D objects of HAT

There are several types of data associates to an integral normal 2D scheme S flat over $\mathbb{Z}$ or $\mathbb{F}_p$ (surface):

$\diamond$ 2D global field: the function field K of S;

$\diamond$ 2D local fields $K_{x,y}$, $x \in y \subset S$, finite separable extensions of $\mathbb{Q}_p((t))$, $\mathbb{R}((t))$, $\mathbb{C}((t))$, $\mathbb{Q}_p\{\{t\}\}$, $\mathbb{F}_p((t_1))((t_2))$;

$\diamond$ 2D (semi-)local-global fields: the function field K_y of the completion of the local ring of a curve $y \subset S$, K_y is a cdvf with global residue field and with a local parameter t_y;

$\diamond$ 2D (semi-)local-global rings K_x, the tensor product of K and the completion $\mathcal{O}_x$ of the local ring of a point $x \in S$.

From these objects one produces 2D geometric adèles $\mathrm{A} \subset \prod \mathrm{K}_{x,y}$, 2D subadèles $\mathrm{B} = \prod \mathrm{K}_y \cap \mathrm{A}$ and 2D subadèles $\mathrm{C} = \prod \mathrm{K}_x \cap \mathrm{A}$.

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Higher adelic theory (HAT) operates with six adelic objects on surfaces:

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\begin{tikzcd}[arrows=dash]
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Geometric adelic structure A is related to rank 1 local integral structure.

Self-duality of its additive group, endowed with appropriate topology, is stronger than Serre duality and it implies the Riemann–Roch theorem on surfaces.

See talks by Czerniawska and Dolce on properties of 2D geometric adèles.

Another analytic/arithmetic adelic structure $\mathbb{A}$ is related to rank 2 local integral structure. It is reflexive and not self-dual.

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HCFT

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A generalisation of Neukirch's CFT mechanism and explicit higher GCFT was produced by F.

HCFT uses Milnor K_n-groups or even better their quotients $K_n^t = K_n / \cap_{m\geq 1} mK_n$

One of key difficulties: for a finite Galois extension L/F of higher fields the homomorphism

$$K_n(F) \to K_n(L)^{G(L/F)}$$

is in general neither injective nor surjective.

2D reciprocity maps: local $\Psi_F\colon K_2^t(F) \longrightarrow G_F^{\mathrm{ab}}$, global $\Psi_K\colon K_2^t(\mathrm{A})/(K_2^t(\mathrm{B})+K_2^t(\mathrm{C})) \longrightarrow G_K^{\mathrm{ab}}$.

All known HCFT are GCFT.

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Zeta functions

The zeta function of a scheme X of finite type over $\mathrm{Spec}(\mathbb{Z})$

$$\zeta_X(s) = \prod_{x \in X_0} (1 - |k(x)|^{-s})^{-1},$$

x runs through closed points of X, $k(x)$ is the finite residue field of x.

The zeta function $\zeta_X(s)$ factorizes into the product of some auxiliary factors and several L-factors or their inverses.

When the function field of X is of characteristic zero and X is two- or higher dimensional, very little is understood about $\zeta_X(s)$.

Remark. One can compare the zeta function to a macro/commutative object and its L-factors to a micro/non-commutative object.

HAT studies zeta functions via higher commutative zeta integrals unlike the study of L-functions via non-commutative 1D zeta integrals in LC.

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Zeta functions of elliptic surfaces

Let E be an elliptic curve over a global field k,
and let $\mathscr{E}$ be a regular model: $\mathscr{E} \to B$ proper flat, where B is the spectrum of the ring of integers of k or a proper smooth curve over a finite field with function field k.

Then

$$\zeta_{\mathscr{E}}(s) = n_{\mathscr{E}}(s)\zeta_E(s), \qquad \zeta_E(s) = \frac{\zeta_k(s)\zeta_k(s-1)}{L_E(s)}.$$

(where

$$n_{\mathscr{E}}(s) = \prod_{b\in B_0, 1\le i\le n_b} (1-|k(b)|^{n_{i,b}(1-s)})^{-1}$$

is the product of zeta functions of affine lines over finite fields, n_b+1 is the number of irreducible componens of the fibre $\mathscr{E}_b$, $n_{i,b}$ are positive integers such that $1+\sum_{1\le i\le n_b} n_{i,b}$ equals the number m_b of irreducible components in the geometric fibre of $\mathscr{E}$ over b.)

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The function $\zeta_E(s)$ was invented by Hasse and is sometimes called the Hasse–Weil zeta function of E, it does not depend on the choice of a model $\mathscr{E}$.

The numerator of $\zeta_E(s)$ is the product of the zeta functions in dimension one.
Its denominator is the L-function of E.

HAT studies the zeta function $\zeta_{\mathscr{E}}$ directly, using commutative 2D methods which universally work over any ground field k.

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Let k be a global field
(number field or function field of a curve over finite field)

The *zeta function*

$$\zeta_k(s) = \sum_{n\geq 1} \frac{a_n}{n^s} = \prod_{\mathfrak{p}} (1 - |k(\mathfrak{p})|^{-s})^{-1}.$$

The completed zeta function

$$\hat{\zeta}_k(s) = \zeta_k(s)\Gamma_k(s)$$

has an integral representation which in its adelic form is

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Using adelic duality and 1D theta-formula (summation formula) Iwasawa and later Tate got

$$\hat{\zeta}_k(s) = \xi(f,s) + \xi(\mathscr{F}(f), 1-s) + \omega(f,s),$$

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The weak (the weakest topology in which every character is continuous) boundary $\partial k^\times = k \setminus k^\times$ is just one point 0 and

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HAT and elliptic curves

Aims of HAT in the case of arithmetic surfaces $\mathscr{E}$:

understand $\zeta_{\mathscr{E}}$ *(and hence* L_E*) via working with a higher zeta integral on 2D adelic spaces using adelic dualities, and then apply to the study of* main open problems about $\zeta_{\mathscr{E}}$. Some of the difficulties:

(1) 2D local fields $K_{x,y}$ are not locally compact spaces, there is no nontrivial real valued translation invariant measure on them,

(2) the structure of $K_2^t(K_{x,y})$ is not known in general.

(3) arithmetic and geometric issues are separated from each other.

Ways to address them:

$\to$ (1) locally compactness is not so important, we can work with $\mathbb{R}((X))$-valued translation invariant measure on $K_{x,y}$ and $K_{x,y}^{\times}$;

$\to$ (2) we can work with $(K_1 \times K_1)(\mathscr{O}_{x,y})$ from which there is a surjective homomorphism to $K_2^t(K_{x,y})$.

$\to$ (3) arithmetic and geometry adelic structures are intertwined at the level of their multiplicative groups and the zeta integral provides a bridge between them.

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(1) 2D local fields $K_{x,y}$ are not locally compact spaces, there is no nontrivial real valued translation invariant measure on them,

(2) the structure of $K_2^t(K_{x,y})$ is not known in general.

(3) arithmetic and geometric issues are separated from each other.

Ways to address them:

$\to$ (1) locally compactness is not so important, we can work with $\mathbb{R}((X))$-valued translation invariant measure on $K_{x,y}$ and $K_{x,y}^{\times}$;

$\to$ (2) we can work with $(K_1 \times K_1)(\mathscr{O}_{x,y})$ from which there is a surjective homomorphism to $K_2^t(K_{x,y})$.

$\to$ (3) arithmetic and geometry adelic structures are intertwined at the level of their multiplicative groups and the zeta integral provides a bridge between them.

HAT and elliptic curves

Aims of HAT in the case of arithmetic surfaces $\mathscr{E}$:

understand $\zeta_{\mathscr{E}}$ *(and hence* L_E*) via working with a higher zeta integral on 2D adelic spaces using adelic dualities, and then apply to the study of* main open problems about $\zeta_{\mathscr{E}}$. Some of the

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Two integral structures of 2D local fields

Let F be a 2D local field whose residue field is a 1D nonarchimedean local field.

Denote by $\mathscr{O}$ the ring of integers of F with respect to its discrete valuation of rank 1 and by t_2 a local parameter of F.

E.g. $\mathscr{O}_{\mathbb{Q}_p((t_2))} = \mathbb{Q}_p[[t_2]]$.

Denote by O the ring of integers with respect to any of its discrete valuations of rank 2.

O equals the preimage of the ring of integers of the residue field.

E.g. $O_{\mathbb{Q}_p((t_2))} = \mathbb{Z}_p + t_2\mathbb{Q}_p[[t_2]]$.

This integral structure O is very much different from the integral structure $\mathscr{O}$.

O is crucial for analysis on 2D local fields and for the study of zeta integrals.

Denote by t_1 a lift of a local parameter of the residue field.

Then $\mathscr{O} = \cup_{j\in\mathbb{Z}} t_1^j O$.

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We have the following 2D picture of O-submodules of F:

$$
\begin{array}{ccccccccc}
\cup_j t_2 t_1^j O = t_2 \mathscr{O} & \cdots & \supset & t_2 t_1^{-1} O & \supset & t_2 O \supset & t_2 t_1 O & \supset \cdots \\
\cup_j t_1^j O = \mathscr{O} & \cdots & \supset & t_1^{-1} O & \supset & O \quad \supset & t_1 O & \supset \cdots \\
\cup_j t_2^{-1} t_1^j O = t_2^{-1} \mathscr{O} & \cdots & \supset & t_2^{-1} t_1^{-1} O & \supset & t_2^{-1} O \supset & t_2^{-1} t_1 O \supset \cdots &
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Measure and integration on 2D local fields

Let $\mathscr{A}$ be the ring of sets generated by distinguished sets $a + t_2^i t_1^j O$.
Define a function

$$\mu(a + t_2^i t_1^j O) = X^i q^{-j}, \quad q = |O : t_1 O|.$$

Theorem

μ is extended to a well defined finitely additive translation invariant *map on $\mathscr{A}$ taking values in $\mathbb{R}((X))$.*

Moreover, for countably many disjoint $A_n \in \mathscr{A}$ such that $\cup A_n \in \mathscr{A}$ and such that $\mu(A_n)$ absolutely converges in $\mathbb{R}((X))$ we get $\mu(A) = \sum \mu(A_n)$.

Unlike the classical case, this measure is not compatible with 2D topology, and various classical methods are not applicable.

This higher Haar measure and integration theory is compatible with the measure and integration on the residue field.

Extensions of this theory to algebraic groups: Morrow (GL_n), Waller (GL_n, SL_n), and a model theoretical work of Hrushovski–Kazhdan in some partial cases.

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Two adelic structures in dimension 2

For a curve y define $\mathscr{O}\mathrm{A}_y$ as a subring of $\prod_{x\in y}\mathscr{O}_{x,y}$ such that for every positive integer r the (x,y)-component is in $\mathscr{O}_x+\mathscr{O}_{x,y}t_y^r$ for almost all closed points x of y.

Define $\mathscr{O}\mathrm{A}_y = \cup_{n\in\mathbb{Z}} t_y^n \mathscr{O}\mathrm{A}_y$.

Define $O\mathrm{A}_y = \mathscr{O}\mathrm{A}_y \cap \prod_{x\in y} O_{x,y}$.

In equal characteristic,

$\mathrm{A}_y, \quad \mathscr{O}\mathrm{A}_y, \quad O\mathrm{A}_y \quad$ can be identified with

$\mathbb{A}_{k(y)}((t_y)), \quad \mathbb{A}_{k(y)}[[t_y]], \quad O\mathbb{A}_{k(y)} + t_y\mathbb{A}_{k(y)}[[t_y]] \quad$ respectively,

where $O\mathbb{A}_{k(y)}$ are integral adèles.

Define geometric adèles A as the restricted product of A_y, for all curves y, with respect to $\mathscr{O}\mathrm{A}_y$.

For all fibres and finitely many horizontal curves of $\mathscr{E}$, define analytic adèles $\mathbb{A}$ as the restricted product of $\mathscr{O}\mathrm{A}_y$, for all curves y, with respect to $O\mathrm{A}_y$.

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2D Fourier transform

Duality. Fix a nontrivial continuous character $\psi\colon F \to \mathbb{C}_1$.
Then every nontrivial continuous character of F is of the form $x \to \psi(ax)$ for some $a \in F$.

For an integrable function f on F define its Fourier transform

$$\mathscr{F}(f) = \int f(\alpha)\psi(\alpha\beta)d\mu(\alpha).$$

Then

$$\mathscr{F}^2(f)(\alpha) = f(-\alpha).$$

Remark. The Fourier transform on 2D local fields $\mathbb{R}((t))$, $\mathbb{C}((t))$ has various features similar to those of the Feynman path integral.

Higher Haar measure, integration and Fourier transform extends from 2D local fields to analytic adèles but not to geometric adèles.

Remark. However, there is a way to run selective integration on geometric adèles, see Czerniawska's talks.

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Local triangle diagramme

In the explicit HCFT a major role is played by a surjective homomorphism

$$\mathfrak{t}_{x,y}\colon \mathcal{O}_{x,y}^{\times}\times\mathcal{O}_{x,y}^{\times}\to K_2^t(K_{x,y}),$$
$$(t_1^i u, t_1^j v)\mapsto (i+j)\{t_1,t_2\}+\{t_1,u\}+\{v,t_2\},\qquad u,v\in O_{x,y}^{\times}.$$

Denote by $VK_2^t(K_{x,y})$ the image of $O_{x,y}^{\times}\times O_{x,y}^{\times}$. We have a commutative diagramme

$$\begin{tikzcd}
\mathcal{O}_{x,y}^{\times}\otimes K_{x,y}^{\times}/\mathcal{O}_{x,y}^{\times} \arrow[d] \arrow[dr] & \\
\mathcal{O}_{x,y}^{\times}\times\mathcal{O}_{x,y}^{\times}/O_{x,y}^{\times} \arrow[r] & K_2^t(K_{x,y})/VK_2^t(K_{x,y}).
\end{tikzcd}$$

The surjective diagonal map is induced by the symbol map; the vertical map sends (α, t_2^m) to $(\alpha^m, 1)$; the composition of the first and second horizontal maps is induced by $\mathfrak{t}_{x,y}$.

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Adelic triangle diagramme

We get the following adelic version of the commutative diagramme above

$$\begin{tikzcd}
\mathbb{A}^\times \otimes \mathrm{A}_{S'}^\times / \mathrm{V}\mathrm{A}_{S'}^\times \arrow[d] \arrow[dr] & \\
\mathbb{A}^\times \times \mathbb{A}^\times / V\mathbb{A}^\times \arrow[r] & K_2^t(\mathrm{A})/\mathrm{V}K_2^t(\mathrm{A}).
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Here $\mathrm{VA}^\times = \mathrm{A}^\times \cap \prod \mathscr{O}_{x,y}^\times$, $V\mathbb{A}^\times = \mathbb{A}^\times \cap \prod O_{x,y}^\times$, and $\mathrm{V}K_2^t(\mathrm{A})$ is the image of $\mathrm{V}\mathbb{A}^\times \times \mathrm{V}\mathbb{A}^\times$.

This diagramme intertwines the multiplicative groups of the adelic structures A and $\mathbb{A}$.

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Zeta integral

The general form of 2D zeta (unramified) integral is

$$\zeta(f, |\,|^s) = \int_{\mathbb{A}^\times \times \mathbb{A}^\times} f(\alpha)\,|\alpha|^s \, d\mu(\alpha)$$

where f is a 2D Bruhat–Schwartz function (such as $\otimes \mathrm{char}_{O_{x,y} \times O_{x,y}}$),
μ is the (appropriately normalised) measure (tensor product of the local measures),
$|\,|$ is the module function associated to μ $\quad (|a| = \mu(aD)/\mu(D))$.

Theorem

On $Re(s) > 2$ the zeta integral $\zeta(f, |\,|^s)$ equals the product of $\zeta_{\mathscr{E}}(s)^2$ times an exponential factor which takes into account the conductor of the model $\mathscr{E}$, times finitely many horizontal zeta integrals.

The zeta integral is a holomorphic function on that half plane.

This theorem essentially gives an integral representation of $\zeta_{\mathscr{E}}(s)^2$.

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2D theta formula

Define analytic adèles $\mathbb{B}$ as the intersection of the product of semi-local-global fields K_y with $\mathbb{A}$.

One gets translation invariant measure and integration on $\mathbb{B}^\times \times \mathbb{B}^\times$ and its weak boundary $\partial\mathbb{B}^\times \times \mathbb{B}^\times$.

These measure are not the lifts of the discrete measures on the product of the function fields of the curves, there are rescaled versions, in some analogy to Tamagawa measure.

Theorem

For a centrally normalized f its transform can be written

$$\mathscr{F}(f)(\alpha) = f(\nu^{-1}\alpha), \qquad |\nu| = 1.$$

We get

$$\int_{\mathbb{B}^\times \times \mathbb{B}^\times} \big(f(\alpha\beta) - |\alpha|^{-1} f(\nu^{-1}\alpha^{-1}\beta)\big)\, d\mu(\beta)$$
$$= \int_{\partial(\mathbb{B}^\times \times \mathbb{B}^\times)} \big(|\alpha|^{-1} f(\nu^{-1}\alpha^{-1}\beta) - f(\alpha\beta)\big)\, d\mu(\beta).$$

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One gets translation invariant measure and integration on $\mathbb{B}^\times \times \mathbb{B}^\times$ and its weak boundary $\partial\mathbb{B}^\times \times \mathbb{B}^\times$.

These measure are not the lifts of the discrete measures on the product of the function fields of the curves, there are rescaled versions, in some analogy to Tamagawa measure.

Theorem

For a centrally normalized f its transform can be written

$$\mathscr{F}(f)(\alpha) = f(\nu^{-1}\alpha), \qquad |\nu| = 1.$$

We get

$$\int_{\mathbb{B}^\times \times \mathbb{B}^\times} \big(f(\alpha\beta) - |\alpha|^{-1} f(\nu^{-1}\alpha^{-1}\beta)\big)\, d\mu(\beta)$$
$$= \int_{\partial(\mathbb{B}^\times \times \mathbb{B}^\times)} \big(|\alpha|^{-1} f(\nu^{-1}\alpha^{-1}\beta) - f(\alpha\beta)\big)\, d\mu(\beta).$$

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Radial coordinates computation of the zeta integral

Using the filtration $\mathbb{A}^\times \times \mathbb{A}^\times > (\mathbb{A}^\times \times \mathbb{A}^\times)_1 > \mathbb{B}^\times \times \mathbb{B}^\times$ where $(\mathbb{A}^\times \times \mathbb{A}^\times)_1$ is the kernel of the module on $\mathbb{A}^\times \times \mathbb{A}^\times$, and the 2D theta formula we obtain

Theorem

On the half plane $Re(s) > 2$ the zeta integral is the sum of three terms

$$\zeta(f, |\,|^s) = \xi(s) + \xi(2-s) + \omega(s).$$

The function $\xi(s)$ extends to an entire function on the complex plane.

The boundary term *(in characteristic 0) is*

$$\omega(s) = \int_0^1 h(x) x^{s-2} dx/x$$

where

$$h(x) = \int_{(\mathbb{A}^\times \times \mathbb{A}^\times)_1 / \mathbb{B}^\times \times \mathbb{B}^\times} \left(\int_{\partial(\mathbb{B}^\times \times \mathbb{B}^\times)} \left(x^2 f(x\gamma\beta) - f(x^{-1}\nu^{-1}\gamma^{-1}\beta)\right) d\mu(\beta) \right) d\mu(\gamma).$$

The function h satisfies $h(x^{-1}) = -x^{-2} h(x)$.

HAT and meromorphic continuation and FE of the zeta function

Which analytic shape should take the function h so that its transform has meromorphic continuation and FE?
For which odd functions their Laplace transform is a symmetric function?

Definition

Let X be a space of complex valued functions on the real line in which the Hahn-Banach theorem holds.

A function $g \in X$ is called *X-mean-periodic* if it satisfies one of the equivalent conditions:

there exists a closed proper linear subspace of X which contains all translates of g;

g is a solution of a homogeneous convolution equation $g * \tau = 0$ where τ is a non-zero element in the dual space of X.

If every translation invariant subspace of X is generated by its finite dimensional translation invariant subspaces, then every mean-periodic function g can be approximated by an appropriately grouped series of exponential polynomials each of which belongs to the closure of the space generated by translations of g. Such series of exponential polynomials generalise Fourier series.

X-mean-periodic functions were studies by Delsarte, Schwartz, Kahane, Lax, Platonov and others.

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Theorem

Assume that the function

$$H(t) = h(e^{-t})$$

is mean-periodic in the space $X_{\exp}$ of smooth functions on $\mathbb{R}$ of exponential growth (when K is of characteristic zero).

Then the boundary term and the zeta integral and hence $\zeta_{\mathscr{E}}(s)$ and $L_E(s)$ have meromorphic continuation and satisfy the functional equation wrt $s \to 2 - s$.

Remark. In a joint work with Ricotta and Suzuki it is shown that if the zeta function $\zeta_{\mathscr{E}}(s)$ extends to a meromorphic function on the complex plane with the expected (conjectural in general) analytic shape and satisfies the functional equation, then $H(t)$ is mean-periodic in the space $X_{\exp}$.

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HAT and GRH

Assuming mean-periodicity of H, the study of the poles of the zeta integral is the study of the Carleman spectrum of H.

Theorem

Assume that H is X_{exp}-mean-periodic.

If the fourth derivative of H keeps its sign near infinity and if the zeta function does not have real poles in the strip $Re(s) \in (1,2)$

then the zeta function does not have complex poles in the same strip.

Remark. Suzuki proved that if L_E has a meromorphic continuation and functional equation, the GRH holds for L_E, and all nonreal zeros of L on the critical line are of multiplicity not greater than $1+$ the multiplicity of the real zero of L at $s=1$, plus some expected technical condition holds, then $H''''(t)$ keeps it sign near infinity.

Remark. Note the fundamental difference with the 1D case. It is easier to study analytically the location of poles (in 2D) than the location of zeros (in 1D).

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HAT and the Tate–BSD conjecture

To compute the local behaviour of $\zeta_{\mathscr{E}}(s)$ at $s=1$ assume that the zeta function has a meromorphic continuation and FE.

Information about $\int_{\partial(\mathbb{B}^\times\times\mathbb{B}^\times)}$ helps to compute the order of the pole of the boundary term $\omega(s)$ (and hence the zeta function) at $s=1$.

Partial information about $\partial(\mathbb{B}^\times\times\mathbb{B}^\times)$ modulo units can be obtained from the adelic triangle diagramme (motivated by explicit HCFT) and the object $\mathbb{B}^\times\otimes \mathrm{B}^\times/(\mathrm{B}^\times\cap \mathrm{VA}^\times)$ in its vertex.

The quotient of $\mathrm{B}^\times/(\mathrm{B}^\times\cap \mathrm{VA}^\times)$ by the image of $K^\times$ and by $p^*\mathrm{Pic}(B)$, where $p\colon \mathscr{E}\to B$, is a finitely generated group with the number of its generators closely related to the rank of $E(k)$.

From the study of geometric adèles related to adelic Riemann–Roch theorem, including topological properties of geometric adèles,

one obtains a factorisation of the boundary term near $s=1$ into the product of finitely many (their number is related to the geometric rank) squares of 1D zeta integrals each of which has a pole of order 1 as $s=1$,

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List of open problems in HAT

List of open problems in HAT is available from

https://ivanfesenko.org/wp-content/uploads/2021/10/prad-1.html