Higher adelic approach to the TBSD conjecture

Ivan Fesenko

1d theory

In dimension one, for global fields, the computation of the adelic zeta integral

$$\hat{\zeta}_{\mathbb{Z}}(s) = \int_{\mathbb{A}_{\mathbb{Q}}^{\times}} f(x)|x|^s d\mu_{\mathbb{A}_{\mathbb{Q}}^{\times}}(x)$$

uses self-duality of the additive group of adeles $\mathbb{A}_{\mathbb{Q}} \simeq X(\mathbb{A}_{\mathbb{Q}})$,

characters $X(\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}) \simeq \mathbb{Q}$,

Fourier transform $\mathscr{F}$ on spaces of functions on adeles,

$$\int_{\mathbb{Q}} g = \int_{\mathbb{Q}} \mathscr{F}(g),$$

radial double integral $\displaystyle \int_{\mathbb{A}_{\mathbb{Q}}^{\times}} = \int_{\mathbb{A}_{\mathbb{Q}}^{\times}/\mathbb{Q}^{\times}} \int_{\mathbb{Q}^{\times}},$

from $\times$ to $+$ $\displaystyle \int_{\mathbb{Q}^{\times}} + \int_{\partial\mathbb{Q}^{\times}} = \int_{\mathbb{Q}}.$

1d theory

The discreteness of global elements $\mathbb{Q}$ in adeles $\mathbb{A}_{\mathbb{Q}}$ and compactness of $\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}$ are associated properties.

In the general case of global fields k the compactness of $\mathbb{A}_k^1/k^\times$ ($\mathbb{A}_k^1$ is the preimage of 1 with respect to $| \, |$) follows from the computation of the zeta integral and it immediately implies the finiteness of the class number.

This computation of the zeta function also implies the Dirichlet's unit theorem.

The Galois group at the background is $\mathrm{Gal}(k^{\mathrm{ab}}/k)$.

Even though it uses objects of class field theory (ideles and idele class group), class field theory is not used in this (1d) Iwasawa–Tate theory.

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1d GL_n theory

For the L-function of an irreducible GL_n-representation of the absolute Galois group G_k of a global field k, its conjectural automorphicity, due to the converse theorems, is closely related to the following conjectural property:

its completed L-function and its twists by appropriate characters, after multiplying with appropriate Gamma-factors, are equal to a zeta integral for an appropriate $M_n(\mathbb{A}_k)$-Bruhat–Schwartz function f:

$$\int_{GL_n(\mathbb{A}_k)} f(\alpha)\, c(\alpha)\, |\det(\alpha)|^s \, d\mu_{GL_n(\mathbb{A}_k)}(\alpha).$$

The additional factor $c(\alpha) = \int_{GL_n(\mathbb{A}_k)^1/GL_n(k)} g_1(\gamma\alpha)\, g_2(\gamma)\, d\mu(\gamma)$ for $n > 1$ involves two cuspidal functions g_i.

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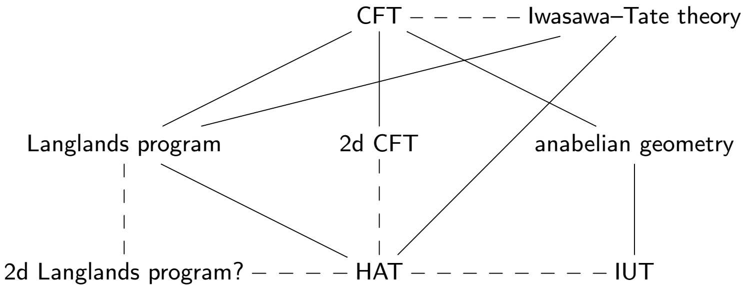
Part of the tree of number theory

CFT = class field theory
HAT = higher adelic theory
2d = two-dimensional (i.e. for arithmetic surfaces)



2d objects of HAT

There are several types of data associated to an integral normal 2d scheme S flat over $\mathbb{Z}$ or $\mathbb{F}_p$ (surface), a closed point x on an irreducible projective curve y on S:

◇ 2d global field: the function field K of S;

◇ 2d local fields/semi-fields: the quotient $K_{x,y}$ of the completion of the localisation of the local ring at x at the local equation of y;

◇ 2d cdvfs for y: the function field K_y of the completion of the local ring of y;

◇ 2d rings for x: the tensor product K_x of K and the completion of the local ring of x.

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Two integral structures in 2d local fields

Let F be a 2d local field whose residue field is a 1d nonarchimedean local field. Denote by $\mathcal{O}$ the ring of integers of F with respect to its discrete valuation of rank 1 and by t_2 a local parameter.

E.g. $\mathcal{O}_{\mathbb{Q}_p((t_2))} = \mathbb{Q}_p[[t_2]]$.

Denote by O the ring of integers with respect to any of its discrete valuations of rank 2. Then O is the preimage of the ring of integers of the residue field. This integral structure O is important for integration on 2d local fields and for 2d zeta integrals.

E.g. $O_{\mathbb{Q}_p((t_2))} = \mathbb{Z}_p + t_2\mathbb{Q}_p[[t_2]]$.

Denote by t_1 a lift of a local parameter of the residue field. Then

$$\mathcal{O} = \cup_{j\in\mathbb{Z}} t_1^j O.$$

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Two integral structures in 2d local fields

We have the following 2d picture of O-submodules of F:

$$
\begin{array}{cccccc}
 & & \cdots\cdots\cdots\cdots & & & \\
t_2\mathscr{O} = \cup_j t_2 t_1^j O & \cdots \supset & t_2 t_1^{-1} O \supset & t_2 O \supset & t_2 t_1 O & \supset \cdots \quad t_2^2\mathscr{O} = \cap_j t_2 t_1^j O \\
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Two adelic structures in dimension 2

For a curve y define $\mathcal{O}\mathrm{A}_y$ as a subring of $\prod_{x\in y}\mathcal{O}_{x,y}$ such that for every positive integer r the (x,y)-component is in $\mathcal{O}_x+\mathcal{O}_{x,y}t_y^r$ for almost all closed points x of y.

Define

$$\mathrm{A}_y=\cup_{n\in\mathbb{Z}}t_y^n\mathcal{O}\mathrm{A}_y,\quad \mathcal{O}\mathrm{A}=\prod\mathcal{O}\mathrm{A}_y,\qquad O\mathrm{A}_y=\mathcal{O}\mathrm{A}_y\cap\prod_{x\in y}O_{x,y}.$$

In equal characteristic,

$\mathrm{A}_y,\quad \mathcal{O}\mathrm{A}_y,\quad O\mathrm{A}_y\quad$ can be identified with

$\mathbb{A}_{k(y)}((t_y)),\quad \mathbb{A}_{k(y)}[[t_y]],\quad O\mathbb{A}_{k(y)}+t_y\mathbb{A}_{k(y)}[[t_y]]\quad$ respectively,

where $O\mathbb{A}_{k(y)}$ are integral adeles.

Then A as the restricted product of A_y, for all curves y, with respect to $\mathcal{O}\mathrm{A}_y$.

For all fibres and a fixed set of finitely many horizontal curves of $\mathscr{E}$, the *second adelic structure* $\mathbb{A}$ is the restricted product of $\mathcal{O}\mathrm{A}_y$ with respect to $O\mathrm{A}_y$.

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Two adelic structures in dimension 2

From these objects one produces

$\mathrm{A} = \prod' K_{x,y}$	complete, local-local
$\mathrm{B} = \prod' K_y = \mathrm{A} \cap \prod K_y$	incomplete, local-global
$\mathrm{C} = \prod' K_x = \mathrm{A} \cap \prod K_x$	incomplete, local-global
K	discrete

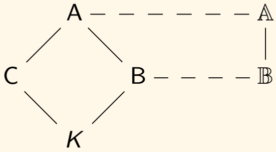
and

$$\mathbb{A},$$

$$\mathbb{B} = \mathbb{A} \cap \prod K_y$$

2d objects of HAT

Higher (2d) adelic theory (HAT) operates with six adelic objects on surfaces:



Geometric adelic structure A is related to rank 1 local integral structure and to algebraic geometry.

Self-duality of its additive group, endowed with appropriate topology, is stronger than Serre duality and it implies the Riemann–Roch theorem on surfaces.

Analytic/arithmetic adelic structure $\mathbb{A}$ is related to rank 2 local integral structure and to 2d zeta integrals.

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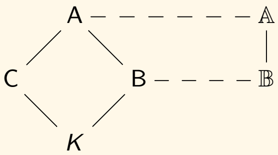

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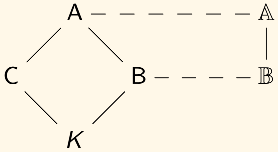

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Zeta functions

The zeta function of a scheme X of finite type over $\mathrm{Spec}(\mathbb{Z})$

$$\zeta_X(s) = \prod_{x \in X_0} (1 - |k(x)|^{-s})^{-1},$$

x runs through closed points of X, $k(x)$ is the finite residue field of x.

The zeta function $\zeta_X(s)$ factorises into the product of some auxiliary factors and several L-factors or their inverses.

When the function field of X is of characteristic zero and X is two- or higher dimensional, very little is understood about $\zeta_X(s)$.

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Zeta functions of elliptic surfaces

Let E be an elliptic curve over a global field k, and let $\mathscr{E}$ be a regular model:

$\mathscr{E} \to B$ proper flat, the generic fibre of $\mathscr{E}$ is E, where B is the spectrum of the ring of integers of k or a proper smooth curve over a finite field with function field k.

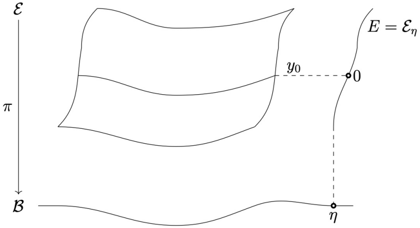
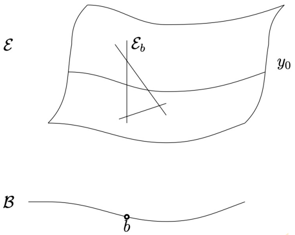


Figure 8.1: the 0 section on arithmetic surface



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$$\zeta_{\mathscr{E}}(s) = n_{\mathscr{E}}(s)\zeta_E(s), \qquad \zeta_E(s) = \frac{\zeta_B(s)\zeta_B(s-1)}{L_E(s)},$$

$$n_{\mathscr{E}}(s) = \prod_{b\in B_0, 1\le i\le n_b} (1-|k(b)|^{n_{i,b}(1-s)})^{-1}$$

where m_b is the number of components in the reduced part of the geometric fibre $\mathscr{E}\times_B k(b)^{\mathrm{sep}}$ of $\mathscr{E}$ over a closed point b of B; so $m_b = 1$ for almost all b.

If $m_b \ne 1$, i.e. the special fibre $\mathscr{E}_b$ is singular, then $n_{i,b}$ are certain positive integers, $1\le i\le n_b$, such that

$$\sum_{1\le i\le n_b} n_{i,b} = m_b - 1,$$

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Zeta functions of elliptic surfaces

The function $\zeta_E(s)$ does not depend on the choice of a model $\mathscr{E}$.

The numerator of $\zeta_E(s)$ is the product of the zeta functions in dimension one.
Its denominator is the L-function of E.

HAT studies the zeta function $\zeta_{\mathscr{E}}$ directly, using commutative 2d methods.

The Galois group at the background is $\mathrm{Gal}(K^{\mathrm{ab}}/K)$, $K = k(\mathscr{E})$ is a 2d global field.

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HAT and elliptic curves

Aims of HAT in the case of arithmetic surfaces $\mathscr{E}$:

understand $\zeta_{\mathscr{E}}$ *(and hence partially* L_E*) via working with a higher zeta integral on 2d adelic spaces using adelic dualities, and then apply to the study of* main open problems about $\zeta_{\mathscr{E}}$.

Some of the difficulties:

(1) 2d local fields $K_{x,y}$ are not locally compact spaces, there is no nontrivial real valued translation invariant measure on them,

(2) unlike 1d, arithmetic and geometric issues are separated from each other in 2d.

Ways to address them:

(1) locally compactness is not so important, we can work with $\mathbb{R}((X))$-valued translation invariant measure on $K_{x,y}$ and $K_{x,y}^{\times}$ discovered in 2001;

(2) arithmetic and geometry adelic structures are intertwined at the level of their multiplicative groups and the zeta integral provides a bridge between them.

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Measure and integration on 2d local fields

Let $\mathscr{A}$ be the ring of sets generated by closed balls with respect to rank 2 discrete valuation $a + t_2^i t_1^j O$.

Define a function

$$\mu(a + t_2^i t_1^j O) = X^i q^{-j}, \quad q = |O : t_1 O|.$$

Theorem

μ is extended to a well defined finitely additive translation invariant *map on $\mathscr{A}$ taking values in $\mathbb{R}((X))$.*

Moreover, for countably many disjoint $A_n \in \mathscr{A}$ such that $\cup A_n \in \mathscr{A}$ and such that $\sum \mu(A_n)$ absolutely converges in $\mathbb{R}((X))$, we get

$$\mu(A) = \sum \mu(A_n).$$

This higher Haar measure and integration theory is compatible with the measure and integration on the residue field, with discrete valuation rk 2 structure, but it is not compatible with topology or metrics.

Measure and integration on 2d local fields

Let $\mathscr{A}$ be the ring of sets generated by closed balls with respect to rank 2 discrete valuation $a + t_2^i t_1^j O$.

Define a function

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Zeta integral

The general form of 2d zeta (unramified) integral is

$$\zeta(f, |\,|^s) = \int_{\mathbb{A}^\times \times \mathbb{A}^\times} f(\alpha)\,|\alpha|^s\, d\mu(\alpha)$$

where f is a 2d Bruhat–Schwartz function (such as $\otimes \mathrm{char}_{O_{x,y} \times O_{x,y}}$),

μ is the (appropriately normalised) measure (tensor product of the local measures),

$|\,|$ is the module function associated to μ $\quad (|a| = \mu(aD)/\mu(D))$,

and the space $\mathbb{A}$ is for a set S' of all vertical curves and a finitely many irreducible regular horizontal curves.

Zeta integral

If $f = \otimes f_y$ then

$$\zeta(f, |\,|^s) = \prod \zeta_y(f_y, |\,|^s).$$

Assume that every singular point of every fibre is a split ordinary double point and the reduction in residual characteristic 2 and 3 is good or multiplicative.

Then for a centrally normalized function f and vertical fibre $y = \mathcal{E}_b$ over $b \in B$

$$\zeta_y(f, |\,|^s) = |k(b)|^{(f_b + m_b - 1)(1-s)} \zeta_y(s)^2$$

where f_b is the conductor, m_b is the number of irreducible geometric components.

For a horizontal curve y the zeta integral $\zeta_y(f, |\,|^s)$ is a meromorphic function satisfying FE with respect to $s \to 2 - s$, holomorphic outside its poles of multiplicity 2 at $s = 0, 2$, $q^s = 1, q^2$.

In characteristic zero on horizontal curve $\zeta_y(f, |\,|^s) = \hat{\zeta}_{k(y)}(s/2)^2$.

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Zeta integral

Theorem (first local computation of the zeta integral)

On $\Re(s) > 2$ the zeta integral $\zeta(f, |\,|^s)$ equals the product of $\zeta_{\mathscr{E}}(s)^2$ times an exponential factor which takes into account the conductor of the model $\mathscr{E}$ and times finitely many horizontal zeta integrals.

Hence the zeta integral is a holomorphic function on that half plane.

The functional equation and meromorphic continuation (the RH) for the zeta integral $\zeta(f, |\,|^s)$, if established, implies the same properties for the zeta function $\zeta_{\mathscr{E}}(s)$ and of the L-function $L_E(s)$.

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The $\mathbb{B}^\times \times \mathbb{B}^\times$

Put

$$T_0 = \mathbb{B}^\times \times \mathbb{B}^\times.$$

Denote by $d_y^{-1/2}$ the cardinality of the multiplicative group of the maximal finite subfield of $k(y)$.

For a function g on $\mathbb{A} \times \mathbb{A}$ such that for almost every fibre y the integral

$$d_y \int g \, d\mu_{(\mathbb{B}_y \times \mathbb{B}_y)^\times} = 1$$

(for example $g = \otimes char_{O_{x,y} \times O_{x,y}}$), define

$$\int_{T_\mathbf{0}} g \, d\mu_{T_\mathbf{0}} = \lim_{S_o} d_{S_o} \prod_{y \in S_o} \int g \, d\mu_{(\mathbb{B}_y \times \mathbb{B}_y)^\times}$$

where d_{S_o} is the product of all d_y attached to vertical fibres in finite $S_o \subset S'$.

So the measure on T_0 is the tensor product of the rescaled fibre measures, and is not the lift of the discrete measure on $k(y)^\times \times k(y)^\times$.

$\int_{\partial T_0}$

The weak boundary of $T_{0,y} = (\mathbb{B}_y \times \mathbb{B}_y)^\times$ is $\mathbb{B}_y \times \mathbb{B}_y \setminus (\mathbb{B}_y \times \mathbb{B}_y)^\times$.

Let ∂T_0 be the weak boundary of T_0, i.e. the union of the product of the weak boundaries of $T_{0,y}$ with y in a finite subset of fibres and horizontal curves and the product of $T_{0,y}$ at all other y.

Define the integral $\int_{\partial T_\mathbf{0}}$ similarly to the definition of $\int_{T_\mathbf{0}}$.

2d theta formula

Theorem

For a centrally normalized f its transform can be written

$$\mathscr{F}(f)(\alpha) = f(\nu^{-1}\alpha), \qquad |\nu| = 1.$$

We get

$$\int_{T_0} \left(f(\alpha\beta) - |\alpha|^{-1} f(\nu^{-1}\alpha^{-1}\beta)\right) d\mu_{T_0}(\beta)$$
$$= \int_{\partial T_0} \left(|\alpha|^{-1} f(\nu^{-1}\alpha^{-1}\beta) - f(\alpha\beta)\right) d\mu_{\partial T_0}(\beta).$$

Applying this theorem, one obtains

Radial computation of the zeta integral

Theorem (second global computation of the zeta integral)

On the half plane $Re(s) > 2$ the zeta integral is the sum of three terms

$$\zeta(f, |\,|^s) = \xi(s) + \xi(2-s) + \omega(s).$$

The function $\xi(s)$ extends to an entire function on the complex plane.

The boundary term *(in characteristic 0) is*

$$\omega(s) = \int_0^1 h(x) x^{s-2} dx/x$$

where

$$h(x) = \int_{(\mathbb{A}^\times \times \mathbb{A}^\times)^{\mathbf{1}}/T_{\mathbf{0}}} \left(\int_{\partial T_{\mathbf{0}}} (x^2 f(x\beta\gamma) - f(\nu^{-1} x^{-1} \beta \gamma^{-1}))\, d\mu(\beta) \right) d\mu(\gamma).$$

The function h satisfies $h(x^{-1}) = -x^{-2} h(x)$.

Mean-periodicity and FE of the zeta function

Definition

Let X be a space of complex valued functions on the real line in which the Hahn-Banach theorem holds.

A function $g \in X$ is called *X-mean-periodic* if it satisfies one of the equivalent conditions:

(i) there exists a closed proper linear subspace of X which contains all translates of g;

(ii) g is a solution of a homogeneous convolution equation $g * \tau = 0$ where τ is a non-zero element in the dual space of X.

HAT and meromorphic continuation and FE of the zeta function

Theorem

Let K be of characteristic 0.
Assume that the function

$$H(t) = h(e^{-t})$$

is mean-periodic in the space of smooth functions on the real line of not more than exponential growth.

Then the boundary term and the zeta integral and hence $\zeta_{\mathscr{E}}(s)$ and $L_E(s)$ have meromorphic continuation and satisfy the functional equation wrt $s \to 2 - s$.

HAT and the poles

Theorem

Maintaining the assumption of mean-periodicity, let in addition the fourth derivative of H keep its sign near infinity.

Then if the zeta function does not have real poles in the strip $Re(s) \in (1,2)$, the zeta function does not have complex poles in the same strip.

Note the fundamental difference with the 1d case. It is easier to study analytically the location of poles in 2d than the location of zeros in 1d.

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HAT and the Tate–BSD conjecture

To compute the local behaviour of $\zeta_{\mathscr{E}}(s)$ at $s=1$ assume that the zeta function has a meromorphic continuation and FE.

Tate reformulated the rank part of the BSD conjecture as the equality

the Euler characteristic of $\mathscr{O}_{\mathscr{E}}^{\times}$ equals to the order of the pole of $\zeta_{\mathscr{E}}(s)$ at $s=1$

HAT reduces the study of the pole of the zeta integral at $s=1$ to the study of the pole of the boundary term at $s=1$.

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HAT and the Tate–BSD conjecture

To relate the arithmetic r and analytic ranks of E, use the previous theory and the fact that geometric adeles know about the Picard rank of $\mathscr{E}$ and the arithmetic rank of $E(k)$.

The group $\mathrm{B}^\times/(\mathrm{B}^\times \cap \mathscr{O}\mathrm{A}^\times)$ is isomorphic to the group of divisors on $\mathscr{E}$, similarly to the one dimensional classical case.

Thus,

$$F_{\mathscr{E}} = \mathrm{coker}\big(K^\times \longrightarrow \mathrm{B}^\times/(\mathrm{B}^\times \cap \mathscr{O}\mathrm{A}^\times)\big)$$

is isomorphic to the Picard group of $\mathscr{E}$.

HAT and the Tate–BSD conjecture

The group $\mathrm{Div}(\mathscr{E})$ is the direct sum of vertical divisors and horizontal divisors, the latter correspond to divisors on its generic fibre.

Choose horizontal curves y_i, $i \in I$, $|I| = r+1$, the images of sections of

$$\pi\colon \mathscr{E} \longrightarrow B$$

which include the image of the zero section and the curves on the surface, corresponding to a choice of free generators of the group $E(\mathrm{k})$.

For every singular fibre $\mathscr{E}_b$ take all the ($k(b)$-rational) components of its reduced part except one which intersects the zero section and denote them by y_j, $1 \le j \le n_b$, where n_b is the number of components of the special fibre $\mathscr{E}_b$ with the component intersecting the zero section excluded.

In addition, choose one nonsingular fibre y_*, and if K is of positive characteristic add it to the above curves.

Denote the whole collection of curves by y_i, $i \in I$, y_j, $j \in J$.

Then $|J| = \sum n_b$ in characteristic zero and $|J| = \sum n_b + 1$ in positive characteristic.

HAT and the Tate–BSD conjecture

In positive characteristic the free part of $\mathrm{NS}(\mathscr{E})$ has rank $r+1+\sum n_b$ (Kodaira, Shioda, Tate).

The group $\pi^*\mathrm{Pic}^0(B)$ is of finite index in the subgroup of divisors numerically equivalent to 0, and the kernel of the natural surjective map from $\mathrm{NS}(\mathscr{E})$ modulo its torsion to $\mathrm{Num}(\mathscr{E})$ is an isomorphism.

Thus, in positive characteristic the group $F_{\mathscr{E}}$ has rank $r+2+\sum n_b$ and its subgroup generated by classes of $\mathrm{B}_{y_i}^{\times}, \mathrm{B}_{y_j}^{\times}$, $i \in I, j \in J$, is of finite index.

In characteristic zero, extending the argument in the positive characteristic, we have a similar description: the subgroup generated by classes of $\mathrm{B}_{y_i}^{\times}, \mathrm{B}_{y_j}^{\times}$, $i \in I, j \in J$, is of finite index in $F_{\mathscr{E}}$ and its rank is $r+1+\sum n_b$.

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Local diagramme

For $x \in y$ the integral ring $\mathcal{O}_{x,y}$ of rank 1 with fraction field $K_{x,y}$, and the integral ring $O_{x,y}$ of rank 2. Then

$$K_{x,y}^{\times} = \mathcal{O}_{x,y}^{\times} \times \langle t_{2x,y} \rangle, \quad \mathcal{O}_{x,y}^{\times} = O_{x,y}^{\times} \times \langle t_{1x,y} \rangle.$$

The following commutative diagramme plays an important role in explicit 2d class field theory

$$\begin{array}{ccc} \mathcal{O}_{x,y}^{\times} \otimes K_{x,y}^{\times}/\mathcal{O}_{x,y}^{\times} & & \\ \downarrow & \searrow & \\ \mathcal{O}_{x,y}^{\times} \times \mathcal{O}_{x,y}^{\times}/O_{x,y}^{\times} & \longrightarrow & K_2^t(K_{x,y})/UK_2^t(K_{x,y}) \end{array}$$

where $UK_2^t(K_{x,y})$ is the kernel of the double boundary map from $K_2^t(K_{x,y})$ to K_1^t and then to K_0, the horizontal map is

$$(t_{1x,y}^{\,i} u, t_{1x,y}^{\,j}) \mapsto (i+j)\{t_{1x,y}, t_{2x,y}\}, \quad u \in O_{x,y}^{\times}$$

the diagonal map is the symbol map

$$\alpha \otimes \beta \mapsto \{\alpha, \beta\},$$

and the vertical continuous map is

$$\alpha \otimes t_{2x,y}^{\,m} \mapsto (\alpha^m, 1).$$

A-diagramme

For a set S' of curves that include all vertical curves and finitely many horizontal curves the previous diagramme implies the adelic commutative diagramme

$$
\begin{array}{ccc}
\mathbb{A}^\times \otimes \mathsf{A}_{S'}^\times/\text{units} & & \\
\downarrow & \searrow & \\
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The diagramme glues together the adelic structures $\mathsf{A}^\times$ and $\mathbb{A}^\times$.

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B-diagramme

The previous diagramme leads to the commutative diagramme

$$\begin{array}{ccccc} & & \mathbb{B}^\times \otimes \mathrm{B}_{S'}^\times/(\mathrm{B}_{S'}^\times \cap \text{units}) & & \\ & & \downarrow & \searrow & \\ T_0 & \longrightarrow & \mathbb{B}^\times \times \mathbb{B}^\times/(\mathbb{B}^\times \cap \text{units}) & \longrightarrow & K_2^t(\mathrm{B})/\big(K_2^t(\mathrm{B}) \cap \text{units}\big). \end{array}$$

HAT and the Tate–BSD conjecture

Using this relation between the two adelic structures and the second computation of the zeta integral, the pole of the boundary term at $s = 1$ comes as the product of the poles of finitely many 1d zeta integrals for y_i and y_j and the pole corresponding to the image of global elements.

One obtains:

The contribution of the image of global elements via $\mathbb{B}^\times \otimes \mathrm{B}^\times/\mathrm{units}$ in the boundary term at $s = 1$ is a non-zero number if and only if the rank part of the Tate–Birch–Swinnerton-Dyer conjecture holds.

The discreteness of global elements in geometric adeles is a crucial property to use.

This discreteness was established in positive characteristic and recently in characteristic zero. This is closely related to the higher adelic Riemann–Roch theorem.

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