Problems in higher adelic theory

Ivan Fesenko

1d theory

To study properties of the Euler–Riemann zeta function

$$\zeta_{\mathbb{Z}}(s) = \sum_{n\geq 1} \frac{1}{n^s} = \prod (1 - p^{-s})^{-1}$$

one can work with the completed zeta function

$$\hat{\zeta}_{\mathbb{Z}}(s) = \pi^{-s/2}\,\Gamma(s/2)\,\zeta_{\mathbb{Z}}(s).$$

1d theory

It has an integral representation

$$\hat{\zeta}_{\mathbb{Z}}(s) = \int_0^{\infty} (\theta(x^2) - 1)\, x^s \,\frac{dx}{x}, \qquad \theta(x) = \sum_{n\in\mathbb{Z}} \exp(-\pi n^2 x).$$

The integral can be rewritten as

$$\int_1^{\infty} (\theta(x^2) - 1)\, x^s \,\frac{dx}{x} + \int_1^{\infty} (\theta(x^2) - 1)\, x^{1-s} \,\frac{dx}{x} + \omega(s)$$

where

$$\omega(s) = \int_0^1 ((\theta(x^2) - 1)x - (\theta(x^{-2}) - 1))\, x^{s-1} \,\frac{dx}{x}.$$

The first two integrals are absolutely convergent and their sum is an entire function on the complex plane symmetric with respect to $s \to 1 - s$.

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The Gauss–Cauchy–Poisson summation formula, which in this case means the functional equation for the theta function $\theta(x^2)x = \theta(x^{-2})$ implies

$$\omega(s) = \int_0^1 (1-x)x^{s-1}\,\frac{dx}{x} = -\left(\frac{1}{s} + \frac{1}{1-s}\right)$$

is a rational function symmetric with respect to $s \to 1-s$, hence the functional equation and meromorphic continuation of the completed Riemann zeta function and the location of its poles follow.

1d theory: revealing more structure

The completed zeta function can be viewed as an adelic zeta integral

$$\hat{\zeta}_{\mathbb{Z}}(s) = \int_{\mathbb{A}_{\mathbb{Q}}^{\times}} f(x)|x|^s d\mu_{\mathbb{A}_{\mathbb{Q}}^{\times}}(x)$$

with respect to an appropriately normalised Haar measure on the group of ideles $\mathbb{A}_{\mathbb{Q}}^{\times}$.

Here $f(x)$ is the tensor product $\otimes f_v(x_v)$ of the characteristic functions $\mathrm{char}_{\mathbb{Z}_p}(x_p)$ and of $\exp(-\pi x_\infty^2)$ at the archimedean place.

f is (almost) an eigenfunction of the appropriately normalised Fourier transform $\mathscr{F}$ on the space of adelic functions.

$|x|$ is the module function associated to μ: $|x| := \mu(xA)/\mu(A)$ for any measurable set A of non-zero volume.

To compare the zeta integral with the completed zeta function one uses the restricted product splitting of ideles:

$$\mathbb{A}_{\mathbb{Q}}^{\times} = \prod{}' \mathbb{Q}_v^{\times}, \quad \int_{\mathbb{A}_{\mathbb{Q}}^{\times}} \otimes g_v = \prod \int_{\mathbb{Q}_v^{\times}} g_v$$

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1d theory

The computation of the zeta integral uses

self-duality of the additive group of adeles $\mathbb{A}_{\mathbb{Q}} \simeq X(\mathbb{A}_{\mathbb{Q}})$,

characters $X(\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}) \simeq \mathbb{Q}$,

Fourier transform $\mathscr{F}$ on spaces of functions on adeles,

$$\int_{\mathbb{Q}} g = \int_{\mathbb{Q}} \mathscr{F}(g),$$

radial double integral $\displaystyle\int_{\mathbb{A}_{\mathbb{Q}}^{\times}} = \int_{\mathbb{A}_{\mathbb{Q}}^{\times}/\mathbb{Q}^{\times}} \int_{\mathbb{Q}^{\times}},$

from $\times$ to $+$ $\displaystyle\int_{\mathbb{Q}^{\times}} + \int_{\partial\mathbb{Q}^{\times}} = \int_{\mathbb{Q}}.$

1d theory

The discreteness of global elements $\mathbb{Q}$ in adeles $\mathbb{A}_{\mathbb{Q}}$ and compactness of $\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}$ are associated properties.

In the general case of global fields k the compactness of $\mathbb{A}_k^1/k^\times$ ($\mathbb{A}_k^1$ is the preimage of 1 with respect to $|\,|$) follows from the computation of the zeta integral and it immediately implies the finiteness of the class number.

This computation of the zeta function also implies the Dirichlet's unit theorem.

The Galois group at the background is $\mathrm{Gal}(k^{\mathrm{ab}}/k)$.

Even though one uses objects of class field theory (ideles and idele class group), class field theory is not used in this (1d) Iwasawa–Tate theory.

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1d GL_n theory

For the L-function of an irreducible GL_n-representation of the absolute Galois group G_k of a global field k, its conjectural automorphicity, due to the converse theorems, is closely related to the following conjectural property:

its completed L-function and its twists by appropriate characters, after multiplying with appropriate Gamma-factors, are equal to a zeta integral for an appropriate $M_n(\mathbb{A}_k)$-Bruhat–Schwartz function f:

$$\int_{GL_n(\mathbb{A}_k)} f(\alpha)\, c(\alpha)\, |\det(\alpha)|^s \, d\mu_{GL_n(\mathbb{A}_k)}(\alpha).$$

The additional factor $c(\alpha) = \int_{GL_n(\mathbb{A}_k)^1/GL_n(k)} g_1(\gamma\alpha)\, g_2(\gamma)\, d\mu(\gamma)$ for $n > 1$ involves two cuspidal functions g_i.

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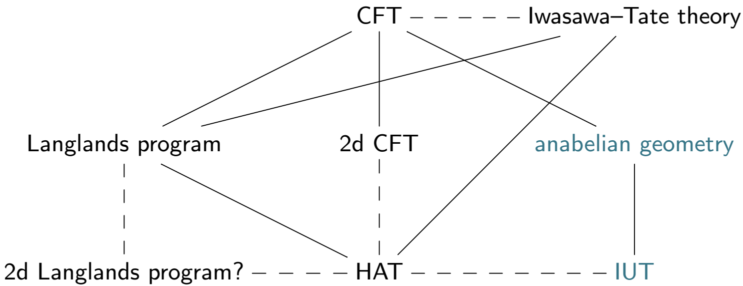
Part of the tree of number theory

CFT = class field theory
HAT = higher adelic theory
2d = two-dimensional (i.e. for arithmetic surfaces)



2d objects of HAT

There are several types of data associated to an integral normal 2d scheme S flat over $\mathbb{Z}$ or $\mathbb{F}_p$ (surface), a closed point x on an irreducible projective curve y on S:

◇ 2d global field: the function field K of S;

◇ 2d local fields/semi-fields: the quotient $K_{x,y}$ of the completion of the localisation of the local ring at x at the local equation of y;

◇ 2d cdvfs for y: the function field K_y of the completion of the local ring of y;

◇ 2d rings for x: the tensor product K_x of K and the completion of the local ring of x.

From these objects one produces

2d geometric adeles $\mathrm{A} \subset \prod \mathrm{K}_{x,y}$,

2d y-subadeles $\mathrm{B} = \prod \mathrm{K}_y \cap \mathrm{A} \subset \prod \mathrm{K}_{x,y}$,

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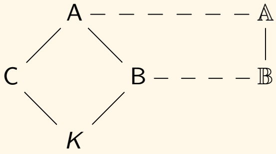
2d objects of HAT

Higher (2d) adelic theory (HAT) operates with six adelic objects on surfaces:



Geometric adelic structure A is related to rank 1 local integral structure and to algebraic geometry.

Self-duality of its additive group, endowed with appropriate topology, is stronger than Serre duality and it implies the Riemann–Roch theorem on surfaces.

Another analytic/arithmetic adelic structure $\mathbb{A}$ is related to rank 2 local integral structure and to 2d zeta integrals.

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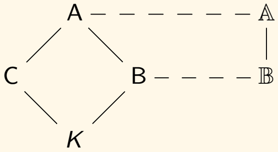

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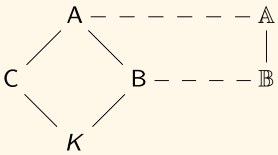

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Zeta functions

The zeta function of a scheme X of finite type over $\mathrm{Spec}(\mathbb{Z})$

$$\zeta_X(s) = \prod_{x \in X_0} (1 - |k(x)|^{-s})^{-1},$$

x runs through closed points of X, $k(x)$ is the finite residue field of x.

The zeta function $\zeta_X(s)$ factorises into the product of some auxiliary factors and several L-factors or their inverses.

When the function field of X is of characteristic zero and X is two- or higher dimensional, very little is understood about $\zeta_X(s)$.

HAT studies zeta functions via higher commutative zeta integrals.

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Zeta functions of elliptic surfaces

Let E be an elliptic curve over a global field k,
and let $\mathscr{E}$ be a regular model:

$\mathscr{E} \to B$ proper flat, where B is the spectrum of the ring of integers of k or a proper smooth curve over a finite field with function field k.

Then

$$\zeta_{\mathscr{E}}(s) = n_{\mathscr{E}}(s)\zeta_E(s), \qquad \zeta_E(s) = \frac{\zeta_B(s)\zeta_B(s-1)}{L_E(s)},$$

$n_{\mathscr{E}}(s)$ is an auxiliary factor that knows about singularities of singular fibres.

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The function $\zeta_E(s)$ does not depend on the choice of a model $\mathscr{E}$.

The numerator of $\zeta_E(s)$ is the product of the zeta functions in dimension one.
Its denominator is the L-function of E.

HAT studies the zeta function $\zeta_{\mathscr{E}}$ directly, using commutative 2d methods.

The Galois group at the background is $\mathrm{Gal}(K^{\mathrm{ab}}/K)$, $K = k(\mathscr{E})$ is a 2d global field.

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HAT and elliptic curves

Aims of HAT in the case of arithmetic surfaces $\mathscr{E}$:

understand $\zeta_{\mathscr{E}}$ *(and hence partially* L_E*) via working with a higher zeta integral on 2d adelic spaces using adelic dualities, and then apply to the study of* main open problems about $\zeta_{\mathscr{E}}$.

Some of the difficulties:

(1) 2d local fields $K_{x,y}$ are not locally compact spaces, there is no nontrivial real valued translation invariant measure on them,

(2) unlike 1d, arithmetic and geometric issues are separated from each other in 2d.

Ways to address them:

(1) locally compactness is not so important, we can work with $\mathbb{R}((X))$-valued translation invariant measure on $K_{x,y}$ and $K_{x,y}^{\times}$ discovered in 2001;

(2) arithmetic and geometry adelic structures are intertwined at the level of their multiplicative groups and the zeta integral provides a bridge between them.

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Zeta integral

The general form of 2d zeta (unramified) integral is

$$\zeta(f, |\,|^s) = \int_{\mathbb{A}^\times \times \mathbb{A}^\times} f(\alpha)\,|\alpha|^s \, d\mu(\alpha)$$

where f is a 2d Bruhat–Schwartz function (such as $\otimes \mathrm{char}_{O_{x,y} \times O_{x,y}}$),
μ is the (appropriately normalised) measure (tensor product of the local measures),
$|\,|$ is the module function associated to μ $\quad (|a| = \mu(aD)/\mu(D))$.

Theorem

On $\mathrm{Re}(s) > 2$ the zeta integral $\zeta(f, |\,|^s)$ equals the product of $\zeta_{\mathscr{E}}(s)^2$ times an auxiliary 1d zeta functions factor.

The zeta integral is a holomorphic function on that half plane.

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Radial computation of the zeta integral

Theorem

On the half plane $Re(s) > 2$ the zeta integral is the sum of three terms

$$\zeta(f, |\,|^s) = \xi(s) + \xi(2-s) + \omega(s).$$

The function $\xi(s)$ extends to an entire function on the complex plane.

The boundary term *(in characteristic 0) is*

$$\omega(s) = \int_0^1 h(x) x^{s-2} dx/x$$

where

$$h(x) = \int_{(\mathbb{A}^\times \times \mathbb{A}^\times)^{\mathbf{1}}/\mathbb{B}^\times \times \mathbb{B}^\times} \left(\int_{\partial(\mathbb{B}^\times \times \mathbb{B}^\times)} (x^2 f(x\beta\gamma) - f(\nu^{-1} x^{-1} \beta \gamma^{-1}))\, d\mu(\beta) \right) d\mu(\gamma).$$

The function h satisfies $h(x^{-1}) = -x^{-2} h(x)$.

Mean-periodicity and FE of the zeta function

Definition

Let X be a space of complex valued functions on the real line in which the Hahn-Banach theorem holds.

A function $g \in X$ is called *X-mean-periodic* if it satisfies one of the equivalent conditions:

(i) there exists a closed proper linear subspace of X which contains all translates of g;

(ii) g is a solution of a homogeneous convolution equation $g * \tau = 0$ where τ is a non-zero element in the dual space of X.

HAT and meromorphic continuation and FE of the zeta function

Theorem

Let K be of characteristic 0.
Assume that the function

$$H(t) = h(e^{-t})$$

is mean-periodic in the space of smooth functions on the real line of not more than exponential growth.

Then the boundary term and the zeta integral and hence $\zeta_{\mathscr{E}}(s)$ and $L_E(s)$ have meromorphic continuation and satisfy the functional equation wrt $s \to 2-s$.

HAT and GRH

Theorem

Maintaining the assumption of mean-periodicity, let in addition the fourth derivative of H keep its sign near infinity.

Then if the zeta function does not have real poles in the strip $Re(s) \in (1, 2)$, the zeta function does not have complex poles in the same strip.

Note the fundamental difference with the 1d case. It is easier to study analytically the location of poles in 2d than the location of zeros in 1d.

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HAT and the Tate–BSD conjecture

To compute the local behaviour of $\zeta_{\mathscr{E}}(s)$ at $s = 1$ assume that the zeta function has a meromorphic continuation and FE.

Information about $\int_{\partial(\mathbb{B}^\times \times \mathbb{B}^\times)}$ helps to compute the order of the pole of the boundary term $\omega(s)$ (and hence the zeta function) at $s = 1$.

Partial information about $\partial(\mathbb{B}^\times \times \mathbb{B}^\times)$ modulo units can be obtained by using the commutative diagram

$$\begin{array}{ccc} \mathbb{B}^\times \otimes \mathrm{B}^\times/\text{units} & & \\ \downarrow & \searrow & \\ \mathbb{B}^\times \times \mathbb{B}^\times/\text{units} & \longrightarrow & K_2^t(\mathrm{B})/\text{units}. \end{array}$$

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HAT and the Tate–BSD conjecture

The quotient of $\mathrm{B}^\times$ modulo the image of units, the image of $K^\times$ and $p^*\mathrm{Pic}(B)$, where $p\colon \mathscr{E} \to B$, is a finitely generated group with the number of its generators equal to the rank of $E(k)$ plus a well known constant.

This leads to a factorisation of the boundary term near $s = 1$ into the product of finitely many (their number is related to the geometric rank) squares of 1d zeta integrals each of which has a pole of order 1 as $s = 1$.

This aims to explain the (conjectured) equality of the analytic rank $\mathrm{ord}_{s=1}\zeta_{\mathscr{E}}(s)$ and the geometric rank $\chi(\mathscr{O}_{\mathscr{E}}^\times)$, i.e. the rank part of the Tate–Birch–Swinnerton-Dyer conjecture.

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Funding and open problems in HAT

List of open problems in HAT:

`https://ivanfesenko.org/wp-content/uploads/prad-1.html`

Selected remaining open problems in 2d adelic analysis and geometry

Ivan Fesenko

2d local aspects

1. Develop a good theory of 2d ramified local zeta integral. Its values will be in R((X)) and the meaning of non-constant terms should be clarified. Use it to construct a new 2d ramification theory. Find its connections with the existing partial approaches to higher ramification theory.

2. Using the theory of translation invariant measure and integration on the additive and multiplicative groups of 2d local fields, develop a theory of translation invariant measure and integration on algebraic groups over 2d local fields. GL(n) case was done by M. Morrow. A new approach more directly generalising the theory on the additive and multiplicative group in Analysis I is done by R. Waller.

3. Develop applications to representation theory of algebraic groups over 2d local fields - some work in this direction is done by R. Waller.

4. Understand better analogies between the Fourier transform on 2d local fields and the Feynman path integral and use these analogies in both directions.

5. Find a general theory that unifies different existing approaches to higher local fields and their arithmetic such as (a) topological and sequential topological, (b) higher translation measure theoretical, (c) iterated ind-pro categories, (d) higher categorical, (e) model theoretical, (f) non-archimedean functional analysis. Some work in direction (f) was done by Alberto Camara.

2d adelic aspects

6. Find a unifying theory that combines features of topological, categorical approaches and model-theoretical approaches, to deal with the two different adelic structures on surfaces: geometric and analytic adelic structures. Find a universal geometric-analytic adelic structure which takes into account the integral structure of rank 1 and of rank 2 and in particular is related to the study of 0-cycles and 1-cycles.

7. Develop explicit global and semi-local-global class field theories for arithmetic surfaces, using the explicit higher local class field theory a la Neukirch, along the description in Analysis II.

8. Develop a 2d adelic approach to Arakelov geometry and the Deligne pairing. Partially done by P. Dolce and W. Czerniawska.

9. Develop a more refined measure and integration which takes into account the range of coefficients of finitely many powers of the main local parameter. Partially done by R. Waller.

10. Using the local and adelic theories for GL_1, and the measure and integration for local GL_n, develop measure and integration on GL_n(A), A the analytic adeles and its application to automorphic representations in 2d. See also 26.

11. Generalising the 1d general linear adelic group theory (e.g. Goldfeld-Hundley), develop appropriate elements of 2 theory. For the 2d local theory, develop 2d analogues of local Whittaker functions, Kirillov model, Jacquet model. See also 26.

12. Develop elements of an enhanced 2d algebraic geometry which takes into account zero cycles and integral structures of rank 2 on surfaces and which could possibly help to find a more universal adelic object which specialises both to geometric adeles and to analytic adeles.

13. Understand 2d theta formula better, from several directions. Try to get an enhanced algebraic geometric proof of 2d theta formula, which can be used to deal with the zeta integral in a more categorical or geometric way. Try to investigate other summation formulas/theta formulas which can be used in the study of the zeta integral.

14. Develop the theory of zeta integrals for singular points on fibres of general type and establish its comparison with the zeta function.

15. Produce a 2d adelic interpretation of the conductor, see also Remark 2 in sect. 40 of Analysis II. Obtain an adelic understanding of the wild part of the conductor.

16. Develop a theory of 2d ramified adelic zeta integral, clarify the meaning of non-free coefficients of ramified zeta integrals and use them to obtain more information about ramification invariants. See also 1.

17. In the context of the correspondence: zeta functions <-> mean-periodic functions, study the mean-periodicity of the boundary function H in the space of smooth functions of exponential growth on the real line, see section 48 of Analysis II and Suzuki-Ricotta-F paper.

18. Develop further the new correspondence zeta functions <-> mean-periodic functions, including connections with the Langlands correspondence. See also 33.

19. Find new applications of mean-periodicity, in particular using Suzuki-Ricotta-F paper and other papers of Masatoshi Suzuki.

20. Progress towards a proof of hypothesis (*) that is closely related to the GRH, in section 51 of Analysis II.

21. Develop the theory sketched in section 55 of Analysis II.

22. Investigate the direction of Remark 1 in section 56 of Analysis II, a 2d generalisation of the Weil-Connes approach to the study of the zeta function and zeta integral and its applications to their meromorphic continuation and functional equation.

23. As part of the study of 2d class field theory, develop further the K_1 times the Brauer group theory for arithmetic surfaces of Shuji Saito and extend it to the general case (without the restriction of absence of real places).

24. Find possible applications of 2d zeta integrals to Langlands correspondences. (this is related to problems of Class field theory, its three main generalisations, and applications).

25. Find possible relations between the two dimensional theta formula and other recent 'non-additive' summation formula on adelic algebraic groups by L. Laffogue.

26. Following the outline in the last section of Adelic approach to zeta functions develop a 2d adelic theory of automorphic functions and representations. See also 30.

27. Using the objects which naturally come from the theory of two dimensional zeta integral understand and develop an enhanced theory of bundles on arithmetic surfaces extending the one-dimensional classical observation of Weil.

28. Various problems on relations between the two dimensional commutative theory of the zeta functions of models of elliptic curves over global fields and one dimensional non-commutative theory for L-factors of the zeta function. Analytically we already have many relations, the issue is to get them algebraically and geometrically.

29. Develop the theory of Eisenstein series on arithmetic surfaces.

30. Various problems on relations between the two dimensional commutative theory of the zeta functions of arithmetic surfaces in positive characteristic and aspects of geometric Langlands correspondence. One analogy between the two theories is that each reduces the analytic aspects of the zeta/L functions to adelic geometric or geometric aspects. See also 18 and 26.

31. In positive characteristic, using Analysis III, find a purely adelic proof of the full BSD conjecture, i.e. without using the previous results proved by other techniques of Tate, Artin and Milne.

32. Following Analysis III, progress towards the relation of the analytic and arithmetic/geometric ranks of a regular model of an elliptic curve over a global field at the central point. Partially done in Analysis III.

33. Develop a general ramification theory for surfaces compatible with 2d CFT and 2d zeta integral, and taking into careful account ramification theory at the one-dimensional residue level. (this is Problem 5 of Class field theory, its three main generalisations, and applications).

34. Develop a special 2d CFT which uses torsion structures, to provide new insights into 2d CFT. (this is Problem 6 of Class field theory, its three main generalisations, and applications).

35. Find new relations between 2d CFT, anabelian geometry and Langlands correspondences. (this is part of Problem 7 of Class field theory, its three main generalisations, and applications).

Surveys and papers

Section K of https://ivanfesenko.org/?page_id=126

including

Adelic approach to the zeta function of arithmetic schemes in dimension two (survey)

https://ivanfesenko.org/wp-content/uploads/2021/10/ada.pdf

Higher adelic theory (talk)

https://ivanfesenko.org/wp-content/uploads/2021/11/hat.pdf