CATEGORICAL VISION FOR QUANTUM THEORY AND ML

Ivan Fesenko

May 14 2022

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MISSED OPPORTUNITIES[1]

BY FREEMAN J. DYSON

It is important for him who wants to discover not to confine himself to one chapter of science, but to keep in touch with various others.

JACQUES HADAMARD

Dyson

'As a working physicist, I am acutely aware of the fact that the marriage between mathematics and physics, which was so enormously fruitful in past centuries, has recently ended in divorce.'

'Twenty years ago ... Richard Feynman gave a description of relativistic quantum field theory in terms of a naive physical picture which he called "sum over histories."

His description seems to make sense as a qualitative guide to the understanding of physical processes, but it makes no sense at all as a mathematical definition.

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Feynman integral

Feynman functional/path integral

$$\int_P \exp\left(\frac{i}{\hbar} S(x)\right) \mathscr{D}x$$

for the action integral $S(x) = \int_0^t \left(\frac{m}{2}\left(\frac{dx}{ds}\right)^2 - V(x(s)\right) ds$ on P

where V is the potential, P is the space of real valued continuous functions on $[0, t]$ with fixed boundary condition.

The problem is with $\mathscr{D}x$ which is a translation invariant measure on P.

The space P is not locally compact, so it does not have a nontrivial translation invariant real valued measure. So the integral does not exist for a mathematician, except some special cases.

Associated alchemistry of renormalisation or regularisation rules, satisfying to most physicists while math rigour is typically missing.

Manin: 'imagine something like the Eiffel Tower, hanging in the air with no foundation, from a mathematical point of view. So it exists and works just right, but standing on nothing we know of. This situation continues to this very day.'

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Problems with foundations of quantum theory

The divorce actually had happenned earlier, in foundations of quantum theory some 100 years ago.

Quantum theory has enormous conceptual problems in its standard formulation that used 100–70 years old mathematics.

These problems are often ignored by many physicists.

Unfortunately, most of mathematics of the last 70 years are typically not known to quantum experts.

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Problems with foundations of quantum theory

There are many alternative interpretations of quantum theory:

◦ The standard Copenhagen interpretation: the state collapses as soon as its degree of macroscopicity becomes so large that we are no longer able to measure the phase between the two terms of the superposition.

The *measurement process*: the quantum object becomes entangled with the macroscopic measurement apparatus and, subsequently, with the experimentalist.

The *instrumental interpretation* of quantum theory that denies the possibility of talking about systems without reference to an external observer.

A 'thing' becomes simply a result of a measurement, physical statements represent our knowledge of events rather than events themselves.

At which point does the superposition collapse into a set of probabilities, a *subjective* phenomenon that only seems to happen when the observer becomes a part of the superposition.

Problems with foundations of quantum theory

◦ Many-worlds interpretation

◦ Consistent histories interpretation

◦ Hidden variables interpretation to deal with assumed incompleteness of the quantum theory formalism

Non-locality and contextuality (Bell, Kochen–Specker) — these features of quantum mechanics are used to obtain quantum advantage over classical computational models in quantum computing.

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Classical theory (quantities are real valued) $\longrightarrow$ a quantisation of it,

but why should quantities be real valued?

Why should quantum probabilities be in $[0,1] \subset \mathbb{R}$?

Frauchiger–Renner: 'Quantum theory cannot consistently describe the use of itself'

Standard mathematics description of quantum theory assumes certain properties of space and/or time but the Planck scale hints otherwise!

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Topos theoretical approach to quantum theory

Isham–Döring's topos theoretical approach to quantum theory

builds locally on (the topos of presheaves of) commutative (hence classical) sub-algebras of the algebra $B(H)$ of all bounded operators on the quantum theory's Hilbert space.

Using topos theory leads to a reformulation of quantum theory which in several aspects looks like classical physics,

propositions can be given truth values without using concepts of measurement or external observer.

See the next talk!

It will be explained below in this talk that the topos theory looks in several aspects like sets theory.

A topos has an internal logical structure that is similar to the way in which Boolean algebra arises in set theory, but instead of two truth values 1 and 0, goes outside Boolean logic with truth values are in a larger set.

Topos theory is a math theory that can 'speak' of indeterminism.

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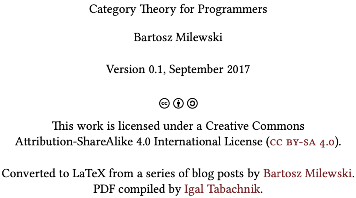
Categories and quantum theory and related theories

There are many texts about categories in quantum theory.


Category Theory for Programmers

Bartosz Milewski

Version 0.1, September 2017

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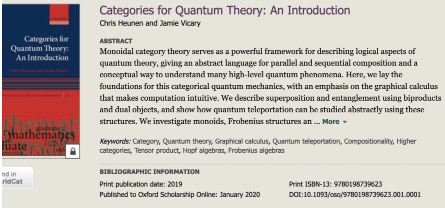
Categories for Quantum Theory: An Introduction

Chris Heunen and Jamie Vicary

ABSTRACT

Monoidal category theory serves as a powerful framework for describing logical aspects of quantum theory, giving an abstract language for parallel and sequential composition and a conceptual way to understand many high-level quantum phenomena. Here, we lay the foundations for this categorical quantum mechanics, with an emphasis on the graphical calculus that makes computation intuitive. We describe superposition and entanglement using biproducts and dual objects, and show how quantum teleportation can be studied abstractly using these structures. We investigate monoids, Frobenius structures an ... More

Keywords: Category, Quantum theory, Graphical calculus, Quantum teleportation, Compositionality, Higher categories, Tensor product, Hopf algebras, Frobenius algebras

BIBLIOGRAPHIC INFORMATION

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There are various theories, somewhere between mathematics and physics, such as string theory or topological quantum field theory or functional quantum field theory, which use categories.

It is interesting which of them are considered as physics by physicists.

Let's look at very basic things at the next 30 slides about sets, the category theory and topos theory in very general terms, emphasising key ideas and concepts, and the associated languages and visions.

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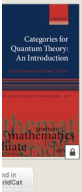

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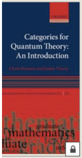

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The vision of categories

Categories formalise certain structures and conceptual frameworks which show up, in different incarnations, in math areas of

- algebraic topology,
- homotopy theory,
- homological algebra,
- number theory,
- various kinds of geometry,
- analysis,
- algebraic K-theory,
- higher class field theory,
- motives,
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Since the mid 1950s Grothendieck further developed category theory and its applications in numerous directions.

Lawvere, 1958: 'I liked experimental physics but did not appreciate the imprecise reasoning in some theoretical courses ... So I decided to study mathematics first ... Categories would clearly be important for simplifying the foundations of continuum physics'

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Categories

Sets can be studied

◦ *internally*, by looking at their subsets, or

◦ *externally*, by looking at the relations to other sets.

A relation of one set to another set is a map f from one set A to another set B. One can represent this map as an arrow $f: A \to B$.

For example,

the image $f(A)$ is a subset of the set B,
the preimage $f^{-1}(b)$ of an element $b \in B$ is a subset of A.

The two studies of sets are closely related to each other.

For example, every subset R of a set S uniquely corresponds to a map f_R from S to the set of two elements $\{1,0\}$:

$$f_R(s) = \begin{cases} 1 & \text{when } s \in R \\ 0 & \text{when } s \notin R \end{cases}$$

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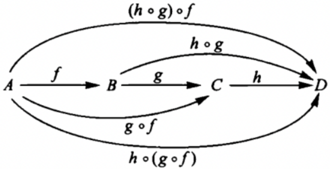
Sets

In the external study, one can compose maps:
for $f: A \to B$ and $g: B \to C$ one has the composite map $g \circ f: A \to C$.

There is the *associativity* property for compositions of maps $f: A \to B$, $g: B \to C$, $h: C \to D$:

$$h \circ (g \circ f) = (h \circ g) \circ f.$$



From sets to categories

For each set A there exists an identity map $1_A \colon A \to A$ which satisfies the (external) properties:

for every map $g \colon A \to B$ we have $g \circ 1_A = g$
and for every map $h \colon C \to A$ we have $1_A \circ h = h$.

For every set A all maps from A to A form a *monoid* $\mathrm{Mor}(A, A)$.

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Categories

A category generalises this.

Definition. A category consists of *objects* $A, B, \ldots$ and *morphisms* (or arrows) between the objects, so that

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Thus, Example 1 of a category is the category **Set** of sets and maps between them.

Categorical point of view is not to pay attention to what the objects and arrows are, but to what patterns of arrows exist between the objects

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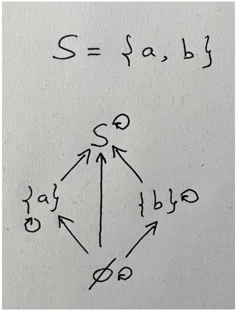
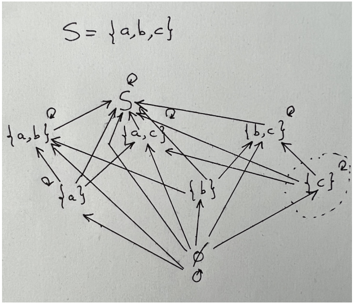
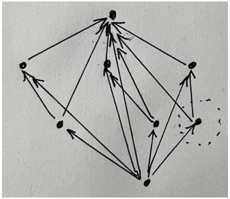
Dots and arrows, to represent categories







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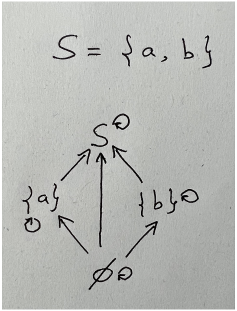

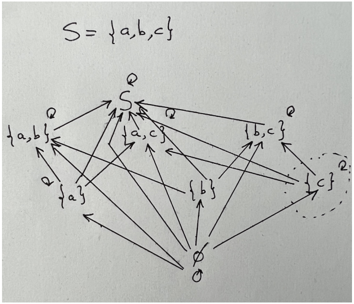

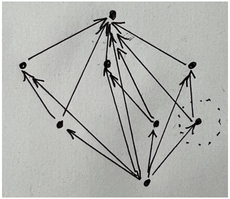

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Categories

Example 2 of a category is in line with internal study, it is

the category $\mathbf{Set}_S$ of all *subsets* R of a given set S and inclusions between them.

We write $R \subseteq S$ for such subsets. Subsets include the empty set $\emptyset$ and the set S.

Morphisms are maps between them, which are inclusions of sets: $R \subseteq T$ with R and T subsets of S.

Subsets are partially ordered with respect to inclusion.

In general, they are not fully ordered since there can be subsets R, T none of which is a subset of the other set.

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In general, they are not fully ordered since there can be subsets R, T none of which is a subset of the other set.

Categories

For two subsets R, T of S we have operations of

intersection $R \cap T$ (it can be called the infimum of R and T) and

union $R \cup T$ (it can be called the supremum of R and T).

They satisfy the *distributive* property:

$$R \cap (T \cup Q) = (R \cap T) \cup (R \cap Q), \qquad R \cup (T \cap Q) = (R \cup T) \cap (R \cup Q).$$

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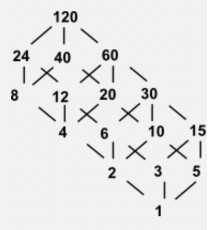
Categories

Every subset R of S also has its complement in S: $S \setminus R$, sometimes denoted $\neg R$, so

$$R \cup \neg R = S, \qquad R \cap \neg R = \emptyset.$$

The lattice of subsets is a *Heyting algebra*.



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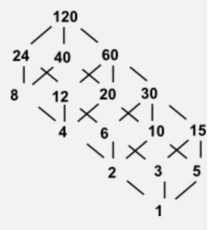

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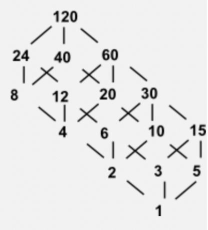

Categories

Example 3 is the category $\mathbf{FSet}_S$ of all finite subsets of a fixed set S and inclusions between them.

In this category the union of infinitely many objects is not necessarily an object of this category, and generally finite subsets do not have their complements in the same category.

Example 4 is the category $\mathbf{ZSet}_S$ of all subsets containing almost all elements of a fixed set S and inclusions between them.

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Suppose that S has a *topology*, i.e. it is a set S with a class of *open subsets*, including S and $\emptyset$, such that

the intersection of any two open subsets is open and

the union of finitely or infinitely many open subsets is open.

In particular, one can work with the

discrete topology on S in which every subset is open, or with

the (Zariski) topology in which open subsets are those which contain almost all elements.

Example 5 is more geometric: the category $\mathbf{Set}_S^{\mathrm{T}}$ of all open subsets of a fixed set S with respect to its topology and inclusions between them.

In this category we can define $\neg U$ as the interior (the maximal open subset) of $S \setminus U$.

Then $U \cup \neg U$ is generally different from S, so no law of excluded middle holds here.

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Example 6 is even more geometric: the category **TSet** of topological spaces with continuous maps as morphisms.

A map $f\colon S \to T$ is called continuous if $f^{-1}(U)$ is open in S for every open U in T.

We can operate with topological spaces using geometric intuition and continuity.

Further examples of categories come from situations when a group K acts on objects, i.e. there are monoid homomorphisms $K \to \mathrm{Mor}(R)$ for every object R.

Many more important examples ...

For every category one has its opposite category which has the same objects and reversed arrows.

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Functors of categories

To compare categories (external study) one needs kind of generalisation of a function. It is called a functor.

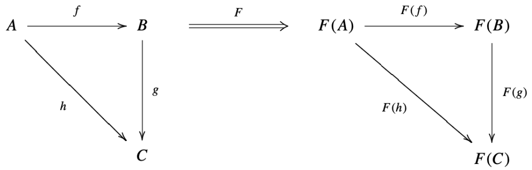
Definition. A (covariant) *functor* F from a category $\mathscr{C}$ to a category $\mathscr{D}$ is

a map which associates to every objects A of $\mathscr{C}$ an object $F(A)$ of $\mathscr{D}$ and

to every morphism $f: A \to B$ in $\mathscr{C}$ a morphism $F(f): F(A) \to F(B)$ in $\mathscr{D}$

such that $F(1_A) = 1_{F(A)}$ and $F(g \circ f) = F(g) \circ F(f)$.

The latter property can be presented by a picture:



Functors of categories

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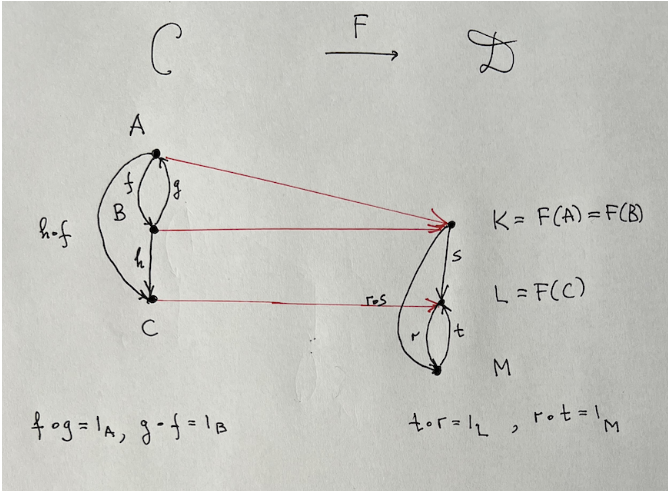

Functors of categories

Contravariant function is similar but reverses arrows and composition, i.e. $F(f)\colon F(B) \to F(A)$ and $F(g \circ f) = F(f) \circ F(g)$

Example of a functor from a category $\mathscr{C}$ to itself: $1_{\mathscr{C}}$ sends each object to itself and each morphism to itself.

Another example is a forgetful functor: e.g. the functor from **TSet** to **Set** which forgets the topology.

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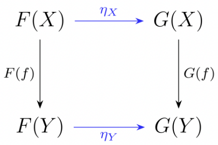
Functors of categories

One also wants to compare functors (external study)

Definition. Let F, G be functors from a category $\mathscr{C}$ to a category $\mathscr{D}$.
A *natural transformation* from F to G is a collection η of morphisms
$\eta_X\colon F(X) \to G(X)$ for every object X of $\mathscr{C}$
such that for every morphism $f\colon X \to Y$ in $\mathscr{C}$ we have

$$\eta_Y \circ F(f) = G(f) \circ \eta_X,$$

i.e. the following *diagram* of morphisms is *commutative*:



Category of functors, equivalence of categories

Definition. A morphism $f\colon A \to B$ of a category $\mathscr{C}$ is called an *isomorphism* if there is a morphism $g\colon B \to A$ such that $g \circ f = 1_A$ and $f \circ g = 1_B$.

In this situation the objects A and B are called *isomorphic*.

Definition. The *category $\mathscr{D}^{\mathscr{C}}$ of functors from $\mathscr{C}$ to $\mathscr{D}$* has objects which are functors and morphisms which are natural transformations between the functors.

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In this situation the functors F and G are called *naturally isomorphic*.

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if there are functors $F\colon \mathscr{C} \to \mathscr{D}$ and $G\colon \mathscr{D} \to \mathscr{C}$ such that

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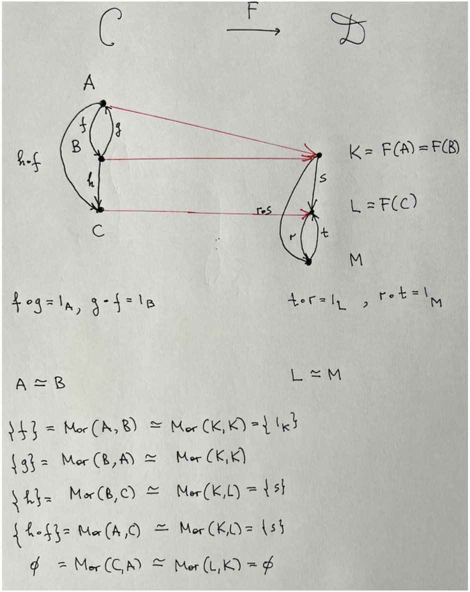
Functors of categories

Example of equivalent categories



Categorical description of sets

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Topos theory

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He realised that many important properties of topological spaces X can be naturally formulated as properties of the categories $\mathrm{Sh}(X)$ of sheaves of sets on the spaces.

At the same time, $X \to \mathrm{Sh}(X)$ is an embedding of continuous structures into categories which are discrete structures.

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The crucial unifying notion of topos is to provide the common geometric intuition for many areas of mathematics and to connect continuous with discrete.

L. Lafforgue: 'The common house of the continuous and the discrete is topos theory'

Grothendieck: 'We can consider that the new geometry is, above all, a synthesis between these two worlds, which until then had been adjoining and closely interdependent, but yet separate: the "arithmetic" world, in which "spaces" without a principle of continuity live, and the world of continuous quantity. In the new vision, these two formerly separate worlds become one.'

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He defined a topos as a certain category of sheaves, by introducing an abstract notion of covering and replacing the topological space X by a site (C, J) consisting of a (small) category C and a Grothendieck's topology (a generalized notion of covering) J on it.

A Grothendieck topos is any category equivalent to the category of sheaves on a site.

A topos has various features similar to the category of sets.

However, unlike sets, the law of excluded middle does not need to hold in a topos.

Statements about a topos are not necessarily either true or false, they can be true somewhere and false somewhere.

Somehow similar to a quantisation of a classical physical theory, constructions in topos theory can often be understood by looking at them in the category of sets or geometrical categories first and then lifting to the general case.

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Definition. *Elementary topos* (or *topos*) is a category with *finite limits* and *colimits*, *exponentials*, and a *subobject classifier*.

In particular, topos has

◦ an initial object (in **Set** this is the empty set),

◦ a terminal object (in **Set** this is the ambient set),

◦ products (in **Set** this is the Cartesian product of two sets) and

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In particular, in topos for every two objects A and B there is an object B^A
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The subject classifier C in **Set** is the two element set $\{1,0\}$ corresponding to true and false.

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◦ **Set**,

◦ K-**Set** where K is a group,

◦ Topos of sheaves on a topological space (in the first approximation, think of functions on open subsets, appropriately glued), for example:

the sheaf of regular functions on a variety

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∘ Topos of presheaves on an arbitrary category $\mathscr{C}$, i.e. the topos of contravariant functors from $\mathscr{C}$ to **Set**, where morphisms are natural transformations between the functors.

∘ Topos of sets living in time: objects are sets $A(t)$ for any time t in the studied interval and maps are $A(t_1) \to A(t_2)$ for $t_1 \leqslant t_2$.

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Lawvere describes objects in a topos as continuously variable sets while classical set theory treats the special case of constant sets.

Topos theory

Each Grothendieck topos is an elementary topos, but the converse property is false: e.g. the category of finite sets, and the category of finite K-sets is an elementary topos but not a Grothendieck topos.

The natural notion of morphism in topos theory is that of geometric morphism. The natural notion of morphism of geometric morphisms if that of geometric transformation.

A global point of a topos T is defined as a geometric morphism from the topos **Set** to the topos T.

There are topoi/toposes which do not have global points.

The non-existence of classical explanations for quantum phenomena somehow corresponds to the non-existence of global sections.

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What matters is the form of a category and its structure.

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Homotopy-theoretic invariants such as the étale fundamental group, playing the key role in anabelian geometry, can be defined as topos invarants.

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Developments

Caramello's recent theory of topos-theoretic 'bridges', which is a general theory of relations between the contents of different mathematical theories.

It uses topos theory to relate and unify mathematics theories and construct 'bridges' between them.

The work of Caramello and L. Lafforgue in using topos theory in various directions

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Modern number theory and quantum theory

Vafa (2000) 'In some sense quantum theory is a bending of physics towards number theory.

However, deep facts of number theory play no role in questions of quantum mechanics...

I predict that in the next century *we will witness deep applications of number theory in fundamental physics* ...

In fact, if I were to guess, I would think that quantum mechanics will be completely reformulated and that number theory will play a key role in this reformulation'

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Even though the Feynman integral does not exist mathematically for potentials or degree higher than 4, there is a rigorous arithmetic analogue of it.

Using structures in higher local fields (and hence some category theory),

a theory of higher translation invariant integral on higher local objects that are not locally compact (e.g. $\mathbb{C}((x))$, a formal loop space) was developed in

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Several analogies have been noticed between ideas of algorithmic anabelian geometry and Mochizuki's IUT theory (they use categories and étale topos) and of quantum theories.

In IUT there are two types of topological monoid structures:
(i) structures coming from Galois groups of symmetries and
(ii) structures coming from 'ordered' objects.
The former structures pass intact through certain processes where the latter structures change.

One can say that the analogues of these two math structures are waves and particles.

The stabiliser formalism in quantum computing uses the facts
(a) the intersection of the Clifford group of a quantum state with the Pauli group still distinquishes quantum states,
(b) this intersection has much smaller number of generators.

Anabelian geometry and IUT work with the stabiliser groups with respect to the Galois action on closed points of relevant arithmetic schemes, they are called decomposition groups.

In anabelian geometry the map from points to conjugacy classes of decomposition groups is the main object of the central Grothendieck section conjecture in anabelian geometry; (a) is the injectivity of the analog of this section map.

Similarly to quantum computing, algorithmic IUT reduces exponential issues to polynomial issues.

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