From two Grothendieck's math legacies to quantum computing and deep neural networks

Ivan Fesenko

Summary

Areas initiated by A. Grothendieck, topos theory and anabelian geometry, remain relatively unknown to AI scientists and quantum computing experts.

There is a potential for the use of some concepts and visions of these math areas in quantum computing, and for understanding of deep neural networks and AI systems.

Categories

Sets can be studied

- *internally*, by looking at their subsets, or
- *externally*, by looking at the relations to other sets.

A relation of one set to another set is a map f from one set A to another set B. One can represent this map as an arrow $f : A \longrightarrow B$.

In internal study, the image $f(A)$ is a subset of the set B, the preimage $f^{-1}(b) = \{a \in A : f(a) = b\}$ of an element $b \in B$ is a subset of A.

The internal and external studies of sets are closely related to each other.

For example, every subset T of a set S uniquely corresponds to a map $char_T$ from S to the set of two elements $\{1,0\}$:

$$char_T(s) = \begin{cases} 1 & \text{when } s \in T \\ 0 & \text{when } s \notin T. \end{cases}$$

Every normal subgroup of a group G correspond to a surjective group homomorphism from G.

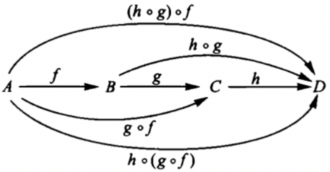
Every ideal of a commutative ring R corresponds to a surjective ring homomorphism from R.

Sets

In the external study, one can compose maps:
for $f: A \longrightarrow B$ and $g: B \longrightarrow C$ to get the composite map $g \circ f: A \longrightarrow C$.

There is the *associativity* property for $f: A \longrightarrow B$, $g: B \longrightarrow C$, $h: C \longrightarrow D$:

$$h \circ (g \circ f) = (h \circ g) \circ f.$$



For each set A there exists an identity map $1_A: A \longrightarrow A$ which satisfies the (external) properties:

for every map $g: A \longrightarrow B$ we have $g \circ 1_A = g$, for every map $h: C \longrightarrow A$ we have $1_A \circ h = h$.

Sets

In the external study, one can compose maps:
for $f: A \longrightarrow B$ and $g: B \longrightarrow C$ to get the composite map $g \circ f: A \longrightarrow C$.

There is the *associativity* property for $f: A \longrightarrow B$, $g: B \longrightarrow C$, $h: C \longrightarrow D$:

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[rrr, bend left=50, "(h \circ g) \circ f"] \arrow[rr, bend right, "g \circ f"'] \arrow[rrr, bend right=50, "h \circ (g \circ f)"'] & B \arrow[r, "g"] \arrow[rr, bend left, "h \circ g"] & C \arrow[r, "h"] & D
\end{tikzcd}$$

For each set A there exists an identity map $1_A: A \longrightarrow A$ which satisfies the (external) properties:

for every map $g: A \longrightarrow B$ we have $g \circ 1_A = g$, for every map $h: C \longrightarrow A$ we have $1_A \circ h = h$.

Categories

A category generalises this.

Definition. A category $\mathscr{C}$ consists of *objects* $A, B, \ldots$ and *morphisms* (or arrows) between the objects, so that

- ∘ for every morphisms $f \colon A \longrightarrow B$ and $g \colon B \longrightarrow C$ one has the composite morphism $g \circ f \colon A \longrightarrow C$
- ∘ the associativity property for morphisms $f \colon A \longrightarrow B$, $g \colon B \longrightarrow C$, $h \colon C \longrightarrow D$ holds:

$$h \circ (g \circ f) = (h \circ g) \circ f$$

- ∘ for every object A there is an identity morphism 1_A (the properties as above).

Categorical point of view is not to pay attention to what the objects and arrows are, but to what *patterns* of arrows exist between the objects

The use of categories in modern mathematics

Categories formalise external properties of certain structures which show up in

- ◦ algebraic topology
- ◦ homological algebra
- ◦ topos theory
- ◦ various cohomology theories including étale cohomology
- ◦ various geometry theories, including derived algebraic geometry
- ◦ algebraic K-theory
- ◦ motives and motivic cohomology
- ◦ higher class field theory
- ◦ higher adelic theory
- ◦ representation theory
- ◦ geometric Langlands correspondences
- ◦ anabelian geometry

Categories is not just a language, it is a new conceptual vision

The use of categories in modern mathematics

Categories formalise external properties of certain structures which show up in

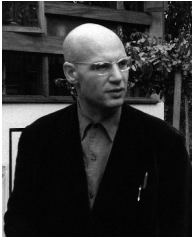
- ◦ algebraic topology
- ◦ homological algebra
- ◦ topos theory
- ◦ various cohomology theories including étale cohomology
- ◦ various geometry theories, including derived algebraic geometry
- ◦ algebraic K-theory
- ◦ motives and motivic cohomology
- ◦ higher class field theory
- ◦ higher adelic theory
- ◦ representation theory
- ◦ geometric Langlands correspondences
- ◦ anabelian geometry

Categories is not just a language, it is a new conceptual vision

Categories

In the 1940s S. Eilenberg and S. Mac Lane developed category theory to provide clearer structural approach to algebraic topology and to build bridges between algebra and topology.

Since the mid 1950s A. Grothendieck further developed category theory and its applications in numerous directions.



Toposes

The notion of topos was introduced in the early sixties by Grothendieck with the original first aim of bringing a topological or geometric intuition also in parts of number theory where actual topological spaces do not occur.

Grothendieck invented topos theory to provide a mathematical underpinning for the missing étale site and étale cohomology theory needed in arithmetic and algebraic geometry and algebra.

He realised that many important properties of topological spaces X can be naturally formulated as properties of the categories $\mathrm{Sh}(X)$ of sheaves of sets on the spaces.

Toposes

The notion of topos was introduced in the early sixties by Grothendieck with the original first aim of bringing a topological or geometric intuition also in parts of number theory where actual topological spaces do not occur.

Grothendieck invented topos theory to provide a mathematical underpinning for the missing étale site and étale cohomology theory needed in arithmetic and algebraic geometry and algebra.

He realised that many important properties of topological spaces X can be naturally formulated as properties of the categories $\mathrm{Sh}(X)$ of sheaves of sets on the spaces.

Toposes

Toposes form a special subclass of categories, which is very suitable for the study of geometric, logical, syntactic, and semantic aspects of the relevant theory.

A topos is the most universal generalisation of a topological space.

At the same time, most geometric intuitions and constructions can be transposed to the new notion of topos.

Somehow similar to a quantisation of a classical physical theory, constructions in topos theory can often be understood by looking at them in the category of sets or geometrical categories first, and then lifting to the general case.

Grothendieck considered his theory as a synthesis between "spaces without continuity principle", i.e. discrete spaces, and "the world of continuous quantity", where "these two once separated worlds become one".

Toposes

Toposes form a special subclass of categories, which is very suitable for the study of geometric, logical, syntactic, and semantic aspects of the relevant theory.

A topos is the most universal generalisation of a topological space.

At the same time, most geometric intuitions and constructions can be transposed to the new notion of topos.

Somehow similar to a quantisation of a classical physical theory, constructions in topos theory can often be understood by looking at them in the category of sets or geometrical categories first, and then lifting to the general case.

Grothendieck considered his theory as a synthesis between "spaces without continuity principle", i.e. discrete spaces, and "the world of continuous quantity", where "these two once separated worlds become one".

Toposes

The crucial unifying notion of topos is to provide the common geometric intuition for many areas of mathematics and to connect continuous with discrete.

$X \to \mathrm{Sh}(X)$ is an embedding of continuous topological space X into a category $\mathrm{Sh}(X)$ which is a discrete structure.

Grothendieck: 'toposes ... is this "deep river", in which come to be married geometry and algebra, topology and arithmetic, mathematical logic and category theory, the world of the continuous and that of the "discontinuous" or "discrete" structures'

Toposes

Grothendieck introduced an abstract notion of covering replacing the topological space X by

a *site* (C, J) consisting of a (small) category C and a Grothendieck's topology (a generalized notion of covering) J on it.

A Grothendieck topos is any category equivalent to the category of sheaves on a site.

A topos has various features similar to the category of sets.

However, unlike sets, the law of excluded middle does not need to hold in a topos.

Statements about a topos are not necessarily either true or false, they can be true somewhere and false somewhere.

Toposes

Grothendieck introduced an abstract notion of covering replacing the topological space X by

a *site* (C, J) consisting of a (small) category C and a Grothendieck's topology (a generalized notion of covering) J on it.

A Grothendieck topos is any category equivalent to the category of sheaves on a site.

A topos has various features similar to the category of sets.

However, unlike sets, the law of excluded middle does not need to hold in a topos.

Statements about a topos are not necessarily either true or false, they can be true somewhere and false somewhere.

Toposes

Definition. An *elementary topos* (or *topos*) is a category with *finite limits* and *colimits*, *exponentials*, and a *subobject classifier*.

In particular, topos has

◦ an initial object, a terminal object,

◦ products and coproducts.

In particular, in topos for every two objects A and B there is an object B^A (in **Set** this is the set of maps from A to B).

The *subject classifier* C in **Set** is the two element set $\{1,0\}$ corresponding to true and false.

In an arbitrary topos, subobjects of A correspond to morphisms from A to C.

Toposes

Definition. An *elementary topos* (or *topos*) is a category with *finite limits* and *colimits*, *exponentials*, and a *subobject classifier*.

In particular, topos has

◦ an initial object, a terminal object,

◦ products and coproducts.

In particular, in topos for every two objects A and B there is an object B^A
(in **Set** this is the set of maps from A to B).

The *subject classifier* C in **Set** is the two element set $\{1,0\}$ corresponding to true and false.

In an arbitrary topos, subobjects of A correspond to morphisms from A to C.

Toposes

Examples:

◦ **Set** corresponding to all subsets of a fixed set

◦ K-**Set** corresponding to all subsets of a given set with an action of a group K

◦ Topos of sheaves on a topological space (in the first approximation, think of functions on open subsets, appropriately glued), for example:

the sheaf of regular functions on a variety

the sheaf of differentiable functions on a differentiable manifold

the sheaf of holomorphic functions on a complex manifold

the sheaf of continuous real-valued functions on any topological space

◦ Topos of presheaves on an arbitrary category $\mathscr{C}$, i.e. the topos of contravariant functors from $\mathscr{C}$ to **Set**, where morphisms are natural transformations between the functors.

Toposes

Examples:

◦ **Set** corresponding to all subsets of a fixed set

◦ K-**Set** corresponding to all subsets of a given set with an action of a group K

◦ Topos of sheaves on a topological space (in the first approximation, think of functions on open subsets, appropriately glued), for example:

the sheaf of regular functions on a variety

the sheaf of differentiable functions on a differentiable manifold

the sheaf of holomorphic functions on a complex manifold

the sheaf of continuous real-valued functions on any topological space

◦ Topos of presheaves on an arbitrary category $\mathscr{C}$, i.e. the topos of contravariant functors from $\mathscr{C}$ to **Set**, where morphisms are natural transformations between the functors.

Toposes

Examples:

○ **Set** corresponding to all subsets of a fixed set

○ K-**Set** corresponding to all subsets of a given set with an action of a group K

○ Topos of sheaves on a topological space (in the first approximation, think of functions on open subsets, appropriately glued), for example:

the sheaf of regular functions on a variety

the sheaf of differentiable functions on a differentiable manifold

the sheaf of holomorphic functions on a complex manifold

the sheaf of continuous real-valued functions on any topological space

○ Topos of presheaves on an arbitrary category $\mathscr{C}$, i.e. the topos of contravariant functors from $\mathscr{C}$ to **Set**, where morphisms are natural transformations between the functors.

Toposes

Each Grothendieck topos is an elementary topos, but the converse property is false.

E.g. the category of finite sets, and the category of finite K-sets is an elementary topos but not a Grothendieck topos.

No quantities show up in category theory and topos theory.

What matters is the form of a category and its structure.

Toposes

Each Grothendieck topos is an elementary topos, but the converse property is false.

E.g. the category of finite sets, and the category of finite K-sets is an elementary topos but not a Grothendieck topos.

No quantities show up in category theory and topos theory.

What matters is the form of a category and its structure.

Toposes and étale objects in arithmetic geometry

The notion of a geometric morphism in topos theory has allowed to build general cohomology theories which cannot be otherwise produced.

The Grothendieck definition of étale sites, étale fundamental group and étale cohomology uses toposes.

For any geometrically integral (quasi-compact) scheme X over a perfect field k one has its étale fundamental group $\pi_1(X)$.

Example. If C is a complex irreducible smooth projective curve minus a finite set of its points, over an algebraically closed field of characteristic 0,

then $\pi_1(C)$ is isomorphic to the profinite completion of the topological fundamental group of the Riemann surface associated to C.

Toposes and étale objects in arithmetic geometry

The notion of a geometric morphism in topos theory has allowed to build general cohomology theories which cannot be otherwise produced.

The Grothendieck definition of étale sites, étale fundamental group and étale cohomology uses toposes.

For any geometrically integral (quasi-compact) scheme X over a perfect field k one has its étale fundamental group $\pi_1(X)$.

Example. If C is a complex irreducible smooth projective curve minus a finite set of its points, over an algebraically closed field of characteristic 0,

then $\pi_1(C)$ is isomorphic to the profinite completion of the topological fundamental group of the Riemann surface associated to C.

Anabelian geometry

A *hyperbolic curve* C over a field k of characteristic zero is a smooth projective geometrically connected curve of genus g minus r points such that the Euler characteristic $2-2g-r$ is negative.

The étale fundamental group of a hyperbolic curve is highly nonabelian, its centre is trivial.

Question 1 (Grothendieck). Are hyperbolic curves over number fields anabelian, i.e. can one restore the curve from its étale fundamental group?

A partial case of Q1 was positively answered by A. Tamagawa and then by S. Mochizuki in the general case.

So, unlike the general case of affine smooth varieties over fields which are determined by their *ring* (two operations) of functions, associated to polynomial equations defining the variety, anabelian curves over number fields are determined by their *group* (one operation) π_1.

This is why anabelian geometry is so powerful.

Anabelian geometry

A *hyperbolic curve* C over a field k of characteristic zero is a smooth projective geometrically connected curve of genus g minus r points such that the Euler characteristic $2-2g-r$ is negative.

The étale fundamental group of a hyperbolic curve is highly nonabelian, its centre is trivial.

Question 1 (Grothendieck). Are hyperbolic curves over number fields anabelian, i.e. can one restore the curve from its étale fundamental group?

A partial case of Q1 was positively answered by A. Tamagawa and then by S. Mochizuki in the general case.

So, unlike the general case of affine smooth varieties over fields which are determined by their *ring* (two operations) of functions, associated to polynomial equations defining the variety, anabelian curves over number fields are determined by their *group* (one operation) π_1.

This is why anabelian geometry is so powerful.

Anabelian geometry

A point x in $X(k)$, i.e. a morphism $\mathrm{Spec}(k) \longrightarrow X$, determines, in a functorial way, a continuous section $G_k \longrightarrow \pi_1(X)$ (well-defined up to composition with an inner automorphism) of the surjective map $\pi_1(X) \longrightarrow G_k$.

Question 2 (Grothendieck). The section conjecture asks if, for a geometrically connected smooth projective curve X over k, of genus > 1, the map from rational points $X(k)$ to the set of conjugacy classes of sections, $x \mapsto D_x = \mathrm{Stab}(x)$, is surjective (injectivity was already known).

Q2 is still unanswered, but various other similar conjectures such as a combinatorial section conjecture are established by Mochizuki and his collaborators.

Anabelian geometry

A point x in $X(k)$, i.e. a morphism $\mathrm{Spec}(k) \longrightarrow X$, determines, in a functorial way, a continuous section $G_k \longrightarrow \pi_1(X)$ (well-defined up to composition with an inner automorphism) of the surjective map $\pi_1(X) \longrightarrow G_k$.

Question 2 (Grothendieck). The section conjecture asks if, for a geometrically connected smooth projective curve X over k, of genus > 1, the map from rational points $X(k)$ to the set of conjugacy classes of sections, $x \mapsto D_x = \mathrm{Stab}(x)$, is surjective (injectivity was already known).

Q2 is still unanswered, but various other similar conjectures such as a combinatorial section conjecture are established by Mochizuki and his collaborators.

Stabiliser formalism in quantum computing

One of the key issues for quantum algorithms is whether they can run in *polynomial time*, instead of *exponential time*. Controlling loss of information/error correction is crucial.

In quantum error correction one uses stabiliser groups in finite dimensional complex spaces.

A map

$$s \mapsto D_s$$

from quantum states s in a $2n$-dimensional vector space over $\mathbb{C}$ to their stabiliser group D_s (unitary matrices acting trivially on s) is *injective*.

However, D_s has too many generators (about 4^n).

Calderbank-Rains-Shor-Sloane, Gottesman, Aaronson-Gottesman (2008) considered the intersection $D_s \cap P_n$.

Here P_n is the group of n-qubit Pauli operators: all tensor products of n Pauli matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and their scalar products with roots of order 4, $|P_n| = 4^{n+1}$.

This intersection $D_s \cap P_n$ has a much smaller number of generators.

Stabiliser formalism in quantum computing

One of the key issues for quantum algorithms is whether they can run in *polynomial time*, instead of *exponential time*. Controlling loss of information/error correction is crucial.

In quantum error correction one uses stabiliser groups in finite dimensional complex spaces.

A map

$$s \mapsto D_s$$

from quantum states s in a $2n$-dimensional vector space over $\mathbb{C}$ to their stabiliser group D_s (unitary matrices acting trivially on s) is *injective*.

However, D_s has too many generators (about 4^n).

Calderbank-Rains-Shor-Sloane, Gottesman, Aaronson-Gottesman (2008) considered the intersection $D_s \cap P_n$.

Here P_n is the group of n-qubit Pauli operators: all tensor products of n Pauli matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and their scalar products with roots of order 4, $|P_n| = 4^{n+1}$.

This intersection $D_s \cap P_n$ has a much smaller number of generators.

Quantum computing

It is easy to show that on the subset of quantum states that are stabilised by exactly 2^n elements of P_n the map

$$s \mapsto D_s \cap P_n$$

is still *injective*.

This subset is further characterised as obtained from $|0\rangle^{\otimes n}$ by CNOT, Hadamard, phase gates only.

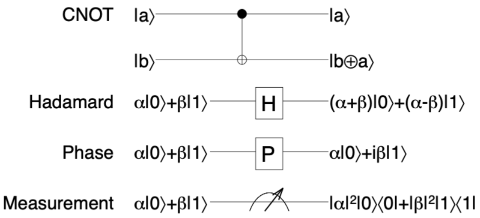


FIG. 1: The four types of gate allowed in the stabilizer formalism

Injectivity of section map

Recently, Y. Hoshi, Mochizuki and S. Tsujimura in their work on the Grothendieck–Teichmüller group obtained further results about the injectivity of the section map

$$x \mapsto D_x \cap G_L$$

from closed points x of hyperbolic curves over number fields to (conjugacy classes of) the intersection of their stabiliser groups (decomposition groups) with the absolute Galois group of various infinite extensions L of the number field k.

Note the similarity with quantum stabiliser theory in quantum computing

Injectivity of section map

Recently, Y. Hoshi, Mochizuki and S. Tsujimura in their work on the Grothendieck–Teichmüller group obtained further results about the injectivity of the section map

$$x \mapsto D_x \cap G_L$$

from closed points x of hyperbolic curves over number fields to (conjugacy classes of) the intersection of their stabiliser groups (decomposition groups) with the absolute Galois group of various infinite extensions L of the number field k.

Note the similarity with quantum stabiliser theory in quantum computing

Section map in arithmetic geometry

The variety of decomposition groups and absolute Galois groups in number theory is much larger than just the Clifford group (the group of unitary matrices that normalise P_n) and P_n in quantum computing.

One perspective is to investigate whether analogues of the decomposition groups and absolute Galois groups in arithmetic geometry will provide new classes of groups useful for quantum computing.

This may allow to go beyond the Clifford ground in quantum computing, an important open challenge in quantum computing.

Topos theory and DNN

According to topos theory, each first-order geometric theory has a classifying topos such that for any other topos, the category of geometrical functors from that other topos to the classifying topos is equivalent to the category of models of the theory in that other topos.

There are different ways of looking at the classifying topos of a theory: logic, geometric, semantic, and syntactic.

Relationships between them naturally arise from invariants of the classifying topos in terms of different sites of definition for it.

Topos theory and DNN

According to topos theory, each first-order geometric theory has a classifying topos such that for any other topos, the category of geometrical functors from that other topos to the classifying topos is equivalent to the category of models of the theory in that other topos.

There are different ways of looking at the classifying topos of a theory: logic, geometric, semantic, and syntactic.

Relationships between them naturally arise from invariants of the classifying topos in terms of different sites of definition for it.

Topos theory and DNN

To an artificial deep neural network (DNN) one can associate an object in a canonical Grothendieck's topos; its learning dynamic corresponds to a flow of morphisms in this topos.

Classifying toposes and their invariants are at the heart of recent developments in applications of topos theory to AI.

Research in this direction has been conducted by L. Lafforgue, O. Caramello, J.-C. Belfiore and D. Bennequin.

Topos theory and DNN

To an artificial deep neural network (DNN) one can associate an object in a canonical Grothendieck's topos; its learning dynamic corresponds to a flow of morphisms in this topos.

Classifying toposes and their invariants are at the heart of recent developments in applications of topos theory to AI.

Research in this direction has been conducted by L. Lafforgue, O. Caramello, J.-C. Belfiore and D. Bennequin.

Topos theory and DNN

There are many invariants of toposes, e.g.

equivalent to a presheaf topos, compact, two-valued, locally connected.

Studying invariants of classifying toposes appropriate for AI models leads to new important links between their syntactic and semantic aspects and a better structural understanding of key issues of the models.

In turn, this can lead to a large range of applications in AI.

Current deep representation learning methods (contrast learning, mask learning, and deep manifold learning) are based on different underlying assumptions and lack theory for a unified description.

The classifying topos may help to reveal the methodological commonalities of various representation learning methods, and based on this, to design new methods that can enhance representation learning.

Topos theory and DNN

There are many invariants of toposes, e.g.

equivalent to a presheaf topos, compact, two-valued, locally connected.

Studying invariants of classifying toposes appropriate for AI models leads to new important links between their syntactic and semantic aspects and a better structural understanding of key issues of the models.

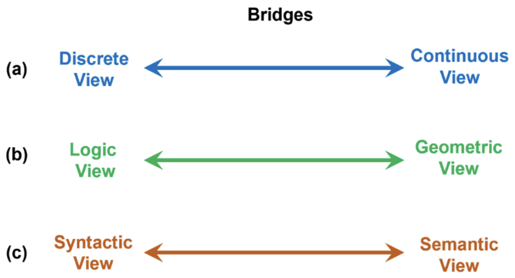
In turn, this can lead to a large range of applications in AI.

Current deep representation learning methods (contrast learning, mask learning, and deep manifold learning) are based on different underlying assumptions and lack theory for a unified description.

The classifying topos may help to reveal the methodological commonalities of various representation learning methods, and based on this, to design new methods that can enhance representation learning.

Topos theory and DNN

Bridges supplied by topos theory



Topos theory and DNN

The theory of sites (C, J), where C is a category and J is a Grothendieck topology on it, simultaneously specialises in the discrete category C and the continuous topology J on it.

The classifying topos can be viewed as the completion of a category C with respect to its Grothendieck topology J.

Topos theory assumes that a (Grothendieck) topology J is available for category C.

However, we are often given unstructured data with an unknown structure or topology.

Develop computational methods to solve this problem based on deep manifold learning is crucial.

This may also bridge between discrete spaces of data and continuous spaces of embedding.

Topos theory and DNN

The theory of sites (C, J), where C is a category and J is a Grothendieck topology on it, simultaneously specialises in the discrete category C and the continuous topology J on it.

The classifying topos can be viewed as the completion of a category C with respect to its Grothendieck topology J.

Topos theory assumes that a (Grothendieck) topology J is available for category C.

However, we are often given unstructured data with an unknown structure or topology.

Develop computational methods to solve this problem based on deep manifold learning is crucial.

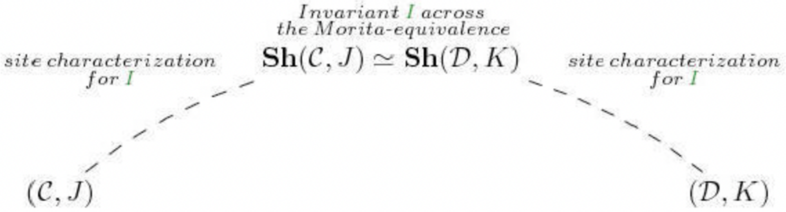
This may also bridge between discrete spaces of data and continuous spaces of embedding.

Topos theory and DNN

Caramello's theory of topos-theoretic 'bridges', which is a general theory of relations between the contents of different mathematical theories.

A Grothendieck topos can be the classifying topos of many geometric logic theories which are Morita-equivalent with each other.

By this transformation, we can not only bridge between logic and geometry but also extract the semantic information that is stable under the changes of syntactic presentation.



Topos theory and DNN

Topos theory provides powerful tools to tackle the challenges of the mathematical foundations of AI.

Topos-theoretic representations can help to build more efficient, reliable, and interpretable deep learning models.

Topos theory has the potential to unify multiple paradigms and techniques in AI, facilitating a more systematic and unified way of developing novel AI techniques.

Topos theory and DNN

More on topos theory and AI:

Jean-Claude Belfiore and Daniel Bennequin, Topos and stacks on deep neural networks, arXiv, 2021

Olivia Caramello, Toposes as 'bridges' for mathematics and artificial intelligence, talk at the Workshop on Semantic Information and Communication: Towards a semantic 6G, Lagrange Center, Paris, March 2023

Olivia Caramello, Syntactic learning via topos theory, talk, New AI theory workshop, November 2023

Laurent Lafforgue, Some possible roles for AI in topos theory, talk at ETH Zurich 2022

Laurent Lafforgue, Grothendieck's topos as mathematics for a Future AI: Illustration by the Problem of Image Representation, talk at Centre Lagrange, October 2023

Laurent Lafforgue, Some sketches for a topos-theoretic AI, talk at Math and Machine learning colloquium series, Barcelona, February 2024