Categories, toposes, anabelian geometry, IUT and quantum computing

Ivan Fesenko

2022-9-12 Crypto-Math CREST

Several general points

the pdf file of this talk

Quantum theory in its standard formulation uses mathematics known 100–70 years ago.

It has enormous foundational problems.

They affect its further developments and applications, including quantum computing and communication, most of which does not use modern mathematics.

Modern arithmetic geometry, including anabelian geometry, and the use of category theory and toposes, remain unknown to quantum people and even to many number theorists.

There is huge potential for the use of some concepts and visions of modern arithmetic geometry in developments of quantum theory, as well as for the use of some novel ideas and concepts of anabelian geometry and IUT in quantum computing

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Relations between mathematics and quantum physics

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BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 78, Number 5, September 1972

MISSED OPPORTUNITIES[1]

BY FREEMAN J. DYSON

It is important for him who wants to discover not to confine himself to one chapter of science, but to keep in touch with various others.
JACQUES HADAMARD

Dyson and Feynman

Dyson (1972): 'As a working physicist, I am acutely aware of the fact that the marriage between mathematics and physics, which was so enormously fruitful in past centuries, has recently ended in divorce.'

'Twenty years ago ... Richard Feynman gave a description of relativistic quantum field theory in terms of a naive physical picture which he called "sum over histories".

His description seems to make sense as a qualitative guide to the understanding of physical processes, but it makes no sense at all as a mathematical definition.'

Feynman (1983): '[math and physics] are very good friends, but they do not consider the same problems, and they do not have the same point of view. The mathematician looks at a very broad area and is interested in everything related to it. The physicist, on the other hand, who is interested in certain specific questions, can go much further in some particular directions...'

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Feynman integral

Feynman functional/path integral

$$\int_P \exp\left(\frac{i}{\hbar} S(x)\right) \mathscr{D}x$$

for the action integral $S(x) = \int_0^t \left(\frac{m}{2} (\frac{dx}{ds})^2 - V(x(s) \right) ds$ on P

where V is the potential, P is the space of real valued continuous functions on $[0, t]$ with fixed boundary condition.

The problem is with $\mathscr{D}x$ which is a translation invariant measure on P:
the space P does not have a nontrivial translation invariant real valued measure.
An attempt by Wiener produces a measure which is not translation invariant.
For potentials of degree $\leqslant 4$ there are rigorous math approaches.

Associated alchemistry of renormalisation or regularisation rules seems satisfying to most physicists while lacking math rigour.

Manin: 'imagine something like the Eiffel Tower, hanging in the air with no foundation, from a mathematical point of view. So it exists and works just right, but standing on nothing we know of. This situation continues to this very day.'

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Modern number theory and quantum theory

There is a rigorous arithmetic analogue of the Feynman integral:

a theory of $\mathbb{R}((t))$-valued higher translation invariant integral on higher local objects that are not locally compact (e.g. $\mathbb{C}((x))$, a formal loop space) was developed in 2001–2004

Measure, integration and elements of harmonic analysis on generalized loop spaces

Ivan Fesenko

Fourier transform in this theory has many similarities to the Feynman integral.

This integration is used to define higher zeta integral whose applications provide entirely new methods to understand several key open problems in number theory

Perspectives of interaction of modern number theory and quantum theory

Vafa (2000) 'In some sense quantum theory is a bending of physics towards number theory.

However, deep facts of number theory play no role in questions of quantum mechanics...

I predict that in the next century we will witness deep applications of number theory in fundamental physics ...

I would think that quantum mechanics will be completely reformulated and that number theory will play a key role in this reformulation'

Belousov (2021) 'Number theory is connected to all other fields – and yet almost none of the related advances utilized in Physics.

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While in many cases related to quantum mechanics & computations – it is perhaps a simpler language to use.'

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Problems with foundations of quantum theory

The divorce between math and physics actually had happened in foundations of quantum theory 100 years ago.

Quantum theory has enormous conceptual problems in its standard formulation that used 100–70 years old mathematics.

These problems are typically ignored by modern physicists.

Unfortunately, most of mathematics of the last 70 years is typically not known to quantum experts.

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Problems with foundations of quantum theory

There are many alternative interpretations of quantum theory:

◦ The standard Copenhagen interpretation: the state collapses as soon as its degree of macroscopicity becomes so large that we are no longer able to measure the phase between the two terms of the superposition.

The *measurement process*: the quantum object becomes entangled with the macroscopic measurement apparatus and, subsequently, with the experimentalist.

The *instrumental interpretation* of quantum theory that denies the possibility of talking about systems without reference to an external observer.

A 'thing' becomes simply a result of a measurement, physical statements represent our knowledge of events rather than events themselves.

At which point does the superposition collapse into a set of probabilities, a *subjective* phenomenon that only seems to happen when the observer becomes a part of the superposition.

Problems with foundations of quantum theory

◦ Many-worlds interpretation

◦ Hidden variables interpretation to deal with assumed incompleteness of the quantum theory formalism

Frauchiger–Renner: 'Quantum theory cannot consistently describe the use of itself'

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Classical theory (quantities are real valued) $\longrightarrow$ a quantisation of it,

but why should quantum quantities be real valued?

Why should quantum probabilities be in $[0,1] \subset \mathbb{R}$?

Standard mathematics description of quantum theory assumes certain properties of space and/or time but the Planck scale hints otherwise!

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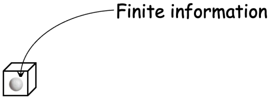
Real numbers and physics

Gisin: 'Real non-rational numbers have no periodicity in their infinitely many decimal digits, but

Finite volume ⇒ finite information

- A finite volume of space can not contain infinitely many bits of information.
- Hence, the position of a classical particle is not a real number.



Mathematical real numbers are not Physically real

real numbers are not really real

Intuitionistic math and physics

Statements about the future can be neither true nor false, but indeterminate.
The law of the excluded middle does not hold.

Relevance of intuitionistic mathematics for physics

Indeterministic Physics	Intuitionist Mathematics
Past, present and future are **not** all given at once	Digits of real numbers are **not** all given at once
Time passes	Numbers are processes
Indeterminism	Numbers contain finite information
The present is thick	The continuum is viscous
The future is open	No law of the excluded middle
Becoming	Choice sequences
Experiencing	Intuitionism

Topos theoretical approach to quantum theory

Non-locality and contextuality (Bell, Kochen–Specker) — these features of quantum mechanics are used to obtain quantum advantage over classical computational models in quantum computing.

Non-Boolean logic in quantum theory

Isham–Döring's topos theoretical approach to quantum theory

builds locally on (the topos of presheaves of) commutative (hence classical) sub-algebras of

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Topos theoretical approach to quantum theory

Using topos theory leads to a reformulation of quantum theory which in several aspects looks like classical physics,

propositions can be given truth values without using concepts of measurement or external observer,

and the logic can be non-Boolean

Topos theory looks in several aspects like sets theory.

A topos has an internal logical structure that is similar to the way in which Boolean algebra arises in set theory, but instead of two truth values 1 and 0, goes outside Boolean logic with truth values are in a larger set.

Topos theory is a math theory that can 'speak' of indeterminism.

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Categories and quantum theory

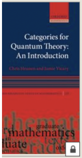
Toposes

Category Theory for Programmers

Bartosz Milewski

Version 0.1, September 2017

Converted to LaTeX from a series of blog posts by Bartosz Milewski.
PDF compiled by Igal Tabachnik.



Categories for Quantum Theory: An Introduction

Chris Heunen and Jamie Vicary

ABSTRACT

Monoidal category theory serves as a powerful framework for describing logical aspects of quantum theory, giving an abstract language for parallel and sequential composition and a conceptual way to understand many high-level quantum phenomena. Here, we lay the foundations for this categorical quantum mechanics, with an emphasis on the graphical calculus that makes computation intuitive. We describe superposition and entanglement using biproducts and dual objects, and show how quantum teleportation can be studied abstractly using these structures. We investigate monoids, Frobenius structures an ... More

Keywords: Category, Quantum theory, Graphical calculus, Quantum teleportation, Compositionality, Higher categories, Tensor product, Hopf algebras, Frobenius algebras

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Print publication date: 2019
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They are not discussed in this talk.

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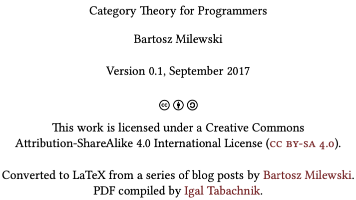
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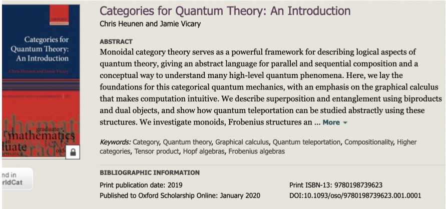
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The use of categories in modern mathematics

Categories formalise certain structures and conceptual frameworks which show up, in different incarnations, in math areas of

- ◦ algebraic topology
- ◦ homological algebra
- ◦ various cohomology theories including étale cohomology
- ◦ various kinds of geometry, including derived algebraic geometry
- ◦ algebraic K-theory
- ◦ motives and motivic cohomology
- ◦ higher class field theory
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- ◦ representation theory
- ◦ geometric Langlands correspondences
- ◦ anabelian geometry
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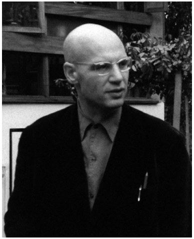
Categories

In the 1940s Eilenberg and Mac Lane developed category theory to provide clearer structural approach to algebraic topology and to build bridges between algebra and topology.

Since the mid 1950s Grothendieck further developed category theory and its applications in numerous directions.



Categorical description of sets

Lawvere, 1958: 'I liked experimental physics but did not appreciate the imprecise reasoning in some theoretical courses ... So I decided to study mathematics first ... Categories would clearly be important for simplifying the foundations of continuum physics'
The elementary theory of the category of sets by Lawvere is an axiomatic formulation of set theory in a category-theoretic spirit.

Lawvere was interested in set theory not be based on membership but on isomorphism-invariant structure and universal mapping properties

He provided a purely categorical description for the category of sets.

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Toposes

The notion of topos was introduced in the early sixties by Grothendieck with the original first aim of bringing a topological or geometric intuition also in parts of number theory where actual topological spaces do not occur.

Grothendieck invented topos theory as part of his approach to prove the Weil conjectures in number theory.

He realised that many important properties of topological spaces X can be naturally formulated as properties of the categories $\mathrm{Sh}(X)$ of sheaves of sets on the spaces.

At the same time, $X \to \mathrm{Sh}(X)$ is an embedding of continuous structures into categories which are discrete structures.

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Toposes

TOPOI

THE CATEGORIAL ANALYSIS OF LOGIC

ROBERT GOLDBLATT

Victoria University of Wellington
New Zealand

Revised edition

1984

NORTH-HOLLAND
AMSTERDAM · NEW YORK · OXFORD

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Toposes

The crucial unifying notion of topos is to provide the common geometric intuition for many areas of mathematics and to connect continuous with discrete.

L. Lafforgue: 'The common house of the continuous and the discrete is topos theory'

Grothendieck: 'We can consider that the new geometry is, above all, a synthesis between these two worlds, which until then had been adjoining and closely interdependent, but yet separate: the "arithmetic" world, in which "spaces" without a principle of continuity live, and the world of continuous quantity. In the new vision, these two formerly separate worlds become one.'

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Grothendieck introduced an abstract notion of covering replacing the topological space X by

a *site* (C, J) consisting of a (small) category C and a Grothendieck's topology (a generalized notion of covering) J on it.

A Grothendieck topos is any category equivalent to the category of sheaves on a site.

A topos has various features similar to the category of sets.

However, unlike sets, the law of excluded middle does not need to hold in a topos.

Statements about a topos are not necessarily either true or false, they can be true somewhere and false somewhere.

Somehow similar to a quantisation of a classical physical theory, constructions in topos theory can often be understood by looking at them in the category of sets or geometrical categories first and then lifting to the general case.

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In particular, topos has

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◦ products (in **Set** this is the Cartesian product of two sets) and

◦ coproducts (in **Set** this is the disjoint union of two sets).

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◦ K-**Set** corresponding to all subsets of a given set with an action of a group K

◦ Topos of sheaves on a topological space (in the first approximation, think of functions on open subsets, appropriately glued), for example:

the sheaf of regular functions on a variety

the sheaf of differentiable functions on a differentiable manifold

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Toposes

Examples of an elementary topos:

○ **Set** corresponding to all subsets of a fixed set

○ K-**Set** corresponding to all subsets of a given set with an action of a group K

○ Topos of sheaves on a topological space (in the first approximation, think of functions on open subsets, appropriately glued), for example:

the sheaf of regular functions on a variety

the sheaf of differentiable functions on a differentiable manifold

the sheaf of holomorphic functions on a complex manifold

the sheaf of continuous real-valued functions on any topological space

Toposes

More examples:

∘ Topos of presheaves on an arbitrary category $\mathscr{C}$, i.e. the topos of contravariant functors from $\mathscr{C}$ to **Set**, where morphisms are natural transformations between the functors.

∘ Topos of sets living in time: objects are sets $A(t)$ for any time t in the studied interval and maps are $A(t_1) \to A(t_2)$ for $t_1 \leqslant t_2$.

Lawvere describes objects in a topos as continuously variable sets while classical set theory treats the special case of constant sets.

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Toposes

Each Grothendieck topos is an elementary topos, but the converse property is false: e.g. the category of finite sets, and the category of finite K-sets is an elementary topos but not a Grothendieck topos.

The natural notion of morphism in topos theory is that of geometric morphism. The natural notion of morphism of geometric morphisms if that of geometric transformation.

A global point of a topos T is defined as a geometric morphism from the topos **Set** to the topos T.

There are toposes which do not have global points.

The non-existence of classical explanations for quantum phenomena somehow corresponds to the non-existence of global points/sections.

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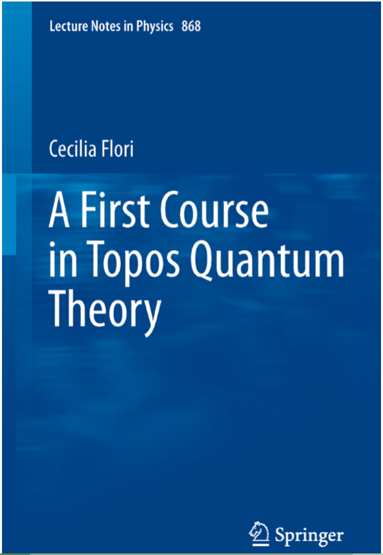
Topos-theoretical approach to quantum theory

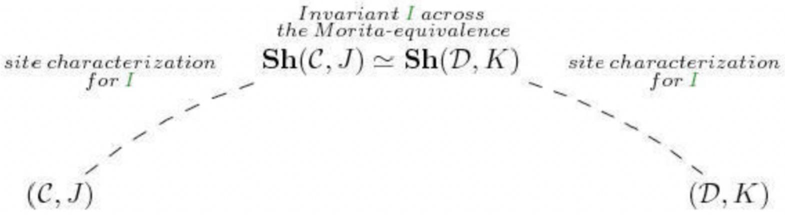
Isham–Döring's topos theoretical approach to quantum theory



Advances in topos theory

Caramello's recent theory of topos-theoretic 'bridges', which is a general theory of relations between the contents of different mathematical theories.

It uses topos theory to relate and unify mathematics theories and construct 'bridges' between them.



Advances in topos theory

The work of O. Caramello and L. Lafforgue in using topos theory in various directions

https://www.oliviacaramello.com/Papers/Papers.htm

https://www.laurentlafforgue.org/publications.html

Applying topos theory in computer science

https://aroundtoposes.com/toposesonline/

New Grothendieck Institute: https://igrothendieck.org/

Toposes and étale objects in arithmetic geometry

No quantities show up in category theory and topos theory.

What matters is the form of a category and its structure.

The notion of a geometric morphism in topos theory has allowed to build general cohomology theories which cannot be otherwise produced.

The Grothendieck definition of étale sites, étale fundamental group and étale cohomology use toposes.

For any geometrically integral (quasi-compact) scheme X over a perfect field k one has its étale fundamental group $\pi_1(X)$.

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Anabelian geometry

Example. If C is a complex irreducible smooth projective curve minus a finite set of its points, over an algebraically closed field of char 0, then $\pi_1(C)$ is isomorphic to the profinite completion of the topological fundamental group of the Riemann surface associated to C.

A *hyperbolic curve* C over a field k of characteristic zero is a smooth projective geometrically connected curve of genus g minus r points such that the Euler characteristic $2-2g-r$ is negative.

The étale fundamental group of a hyperbolic curve is highly nonabelian, its centre is trivial.

Question 1 (Grothendieck). Are hyperbolic curves over number fields anabelian, i.e. can one restore the curve from its étale fundamental group?

A partial case of Q1 was positively answered by A. Tamagawa and then by S. Mochizuki in the general case.

So, unlike the general case of affine smooth varieties over fields which are determined by their *ring* (two operations) of functions, associated to polynomial equations defining the variety, anabelian curves over number fields are determined by their *group* (one operation) π_1.
This is why anabelian geometry is so powerful.

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A point x in $X(k)$, i.e. a morphism $\mathrm{Spec}(k) \longrightarrow X$, determines, in a functorial way, a continuous section $G_k \longrightarrow \pi_1(X)$ (well-defined up to composition with an inner automorphism) of the surjective map $\pi_1(X) \longrightarrow G_k$.

Question 2 (Grothendieck). The section conjecture asks if, for a geometrically connected smooth projective curve X over k, of genus > 1, the map from rational points $X(k)$ to the set of conjugacy classes of sections, $x \mapsto D_x = \mathrm{Stab}(x)$, is surjective (injectivity was already known).

Q2 is still unanswered, but various other similar conjectures such as a combinatorial section conjecture are established by S. Mochizuki and his collaborators.

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Stabiliser formalism in quantum computing

One of the key issues for quantum algorithms is whether they can run in *polynomial time*, instead of *exponential time*. Controlling loss of information/error correction is crucial.

In quantum error correction one uses stabiliser groups in finite dimensional complex spaces.

A map

$$s \mapsto D_s$$

from quantum states s in a $2n$-dimensional vector space over $\mathbb{C}$ to their stabiliser group D_s (unitary matrices acting trivially on s) is *injective*.

However, D_s has too many generators (about 4^n).

Calderbank-Rains-Shor-Sloane, Gottesman, Aaronson-Gottesman (2008) considered the intersection $D_s \cap P_n$.

Here P_n is the group of n-qubit Pauli operators: all tensor products of n Pauli matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and their scalar products with roots of order 4, $|P_n| = 4^{n+1}$.

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Quantum computing

It is easy to show that on the subset of quantum states that are stabilised by exactly 2^n elements of P_n the map

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is still *injective*.

This subset is further characterised as obtained from $|0\rangle^{\otimes n}$ by CNOT, Hadamard, phase gates only.

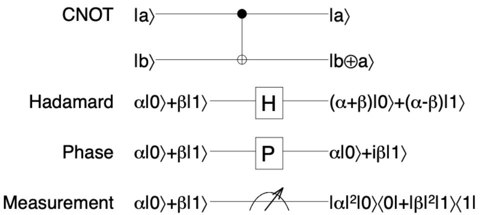

FIG. 1: The four types of gate allowed in the stabilizer formalism

Injectivity of section map

Recently, Hoshi, Mochizuki and Tsujimura in their work on the Grothendieck–Teichmueller group obtained further results about the injectivity of the section map

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Section map in arithmetic geometry

The variety of decomposition groups and absolute Galois groups in number theory is much larger than just the Clifford group (the group of unitary matrices that normalise P_n) and P_n in quantum computing.

One perspective is to investigate whether analogues of the decomposition groups and absolute Galois groups in arithmetic geometry will provide new classes of groups useful for quantum computing.

This may allow to go beyond the Clifford ground in quantum computing, an important open challenge in quantum computing.

At the same time, there is a substantial difference between profinite decomposition groups and discrete stabiliser groups in quantum computing: the centre of the former is trivial while the centre of Clifford group is infinite but its quotient group by its centre is finite.

However, when one works with those arithmetic stabiliser groups, often one considers them as the projective limit of their quotients which are extensions of a finite group by infinite abelian and such quotients modulo their centre are finite groups as well.

In quantum communication, quantum entanglement and violations of Bell's inequality is about elements of the tensor product which are not tensors themselves.

So far quantum computing has not used mathematics of the last 50 years.

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IUT papers

Publ. RIMS Kyoto Univ. **57** (2021), 3–207
DOI 10.4171/PRIMS/57-1-1

Inter-universal Teichmüller Theory I: Construction of Hodge Theaters

by

Shinichi MOCHIZUKI

Abstract

The present paper is the first in a series of four papers, the goal of which is to establish an *arithmetic* version of *Teichmüller theory* for **number fields** equipped with an **elliptic curve** – which we refer to as "**inter-universal Teichmüller theory**" – by applying the theory of *semi-graphs of anabelioids*, *Frobenioids*, the *étale theta function*, and *log-shells* developed in earlier papers by the author. We begin by fixing what we call "*initial* Θ-*data*", which consists of an *elliptic curve* E_F over a *number field* F, and a *prime number* $l \geq 5$, as well as some other technical data satisfying certain technical properties. This data determines various *hyperbolic orbicurves* that are related via finite étale coverings to the once-punctured elliptic curve X_F determined by E_F. These finite étale coverings admit various *symmetry properties* arising from the **additive** and **multiplicative** structures on the ring $\mathbb{F}_l = \mathbb{Z}/l\mathbb{Z}$ acting on the *l-torsion points* of the elliptic curve. We then construct "$\Theta^{\pm\mathrm{ell}}$NF-*Hodge theaters*" associated to the given Θ-data. These $\Theta^{\pm\mathrm{ell}}$NF-Hodge theaters may be thought of as *miniature models of* ***conventional scheme theory*** in which the **two underlying combinatorial dimensions** of a number field – which may be thought of as corresponding to the **additive** and **multiplicative** structures of a ring or, alternatively, to the **group of units** and **value group** of a local field associated to the number field – are, in some sense, "**dismantled**" or "**disentangled**" from one another. All $\Theta^{\pm\mathrm{ell}}$NF-Hodge theaters are isomorphic to one another, but may also be related to one another by means of a "**Θ-link**", which relates certain *Frobenioid-theoretic* portions of one $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater to another in a fashion that is ***not compatible*** *with the respective* ***conventional ring/scheme theory structures***. In particular, it is a *highly nontrivial problem to relate the ring structures* on either side of the Θ-link to one another. This will be achieved, up to certain "*relatively mild indeterminacies*", in future papers in the series by applying the **absolute anabelian geometry** developed in earlier papers by the author. The resulting *description of an* "***alien ring structure***" [associated, say, to the *domain* of the Θ-link] in terms of a given ring structure [associated, say, to the *codomain* of the Θ-link] will be applied in the final paper of the series

Publ. RIMS Kyoto Univ. **57** (2021), 627–723
DOI 10.4171/PRIMS/57-1-4

Inter-universal Teichmüller Theory IV: Log-Volume Computations and Set-Theoretic Foundations

by

Shinichi MOCHIZUKI

Abstract

The present paper forms the fourth and final paper in a series of papers concerning "**inter-universal Teichmüller theory**". In the first three papers of the series, we introduced and studied the theory surrounding the **log-theta-lattice**, a *highly noncommutative* two-dimensional diagram of "*miniature models of conventional scheme theory*", called $\Theta^{\pm\mathrm{ell}}$NF-*Hodge theaters*, that were associated, in the first paper of the series, to certain data, called *initial* Θ-*data*. This data includes an *elliptic curve* E_F over a *number field* F, together with a *prime number* $l \geq 5$. Consideration of various properties of the log-theta-lattice led naturally to the establishment, in the third paper of the series, of **multiradial algorithms** for constructing "**splitting monoids of LGP-monoids**". Here, we recall that "multiradial algorithms" are algorithms that make sense from the point of view of an "**alien arithmetic holomorphic structure**", i.e., the ring/scheme structure of a $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater related to a given $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater by means of a *non-ring/scheme-theoretic* horizontal arrow of the log-theta-lattice. In the present paper, estimates arising from these multiradial algorithms for splitting monoids of LGP-monoids are applied to verify various **diophantine results** which imply, for instance, the so-called **Vojta Conjecture** for hyperbolic curves, the **ABC Conjecture**, and the **Szpiro Conjecture** for elliptic curves. Finally, we examine – albeit from an extremely *naive/non-expert* point of view! – the *foundational/set-theoretic* issues surrounding the *vertical* and *horizontal arrows* of the log-theta-lattice by introducing and studying the basic properties of the notion of a "**species**", which may be thought of as a sort of formalization, via set-theoretic formulas, of the intuitive notion of a "*type of mathematical object*". These foundational issues are closely related to the central role played in the present series of papers by various results from **absolute anabelian geometry**, as well as to the idea of ***gluing together*** *distinct models of conventional scheme theory*, i.e., in a fashion that lies outside the framework of conventional scheme theory. Moreover, it is

IUT papers

For surveys of IUT see

The mathematics of mutually alien copies: from gaussian integrals to inter-universal Teichmüller theory, Inter-universal Teichmüller Theory Summit 2016,RIMS Kokyuroku Bessatsu B84, Res. Inst. Math. Sci. (RIMS), Kyoto (2021) 23-192, by Shinichi Mochizuki

On the essential logical structure of inter-universal Teichmüller theory in terms of logical 'and' logical 'or' relations: Report on the occasion of the publication of the four main papers on inter-universal Teichmüller theory, by Shinichi Mochizuki

Arithmetic deformation theory via arithmetic fundamental groups and nonarchimedean theta functions, notes on the work of Shinichi Mochizuki, Europ. J. Math. (2015) 1:405-440, by Ivan Fesenko

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EE-IUT paper

S. MOCHIZUKI, I. FESENKO, Y. HOSHI, A. MINAMIDE AND W. POROWSKI
KODAI MATH. J.
45 (2022), 175–236

EXPLICIT ESTIMATES IN INTER-UNIVERSAL TEICHMÜLLER THEORY

Shinichi Mochizuki, Ivan Fesenko, Yuichiro Hoshi, Arata Minamide and Wojciech Porowski

For every two coprime (i.e. no common prime divisors) positive integer numbers a, b

$$\log(ab(a+b)) < 6 \log \mathrm{rad}(ab(a+b)) \qquad \text{if} \qquad \log(ab(a+b)) > 1.7 \cdot 10^{30}$$

where the radical rad of a number is the product of all distinct prime numbers dividing it.

For example, this effective abc inequality implies that for all sufficiently large m the number $2^m + 3^m$ is divisible by (effectively computable) large prime numbers whose power in the factorisation of $2^m + 3^m$ does not exceed 5. This is a new way to find very large prime numbers.

On IUT on two pages

Algebraic geometry involves locally the correspondence between affine varieties and commutative rings.

Anabelian geometry for hyperbolic curves over number fields and other fields is a correspondence between these geometric objects and their arithmetic fundamental groups (or slightly more complicated objects).

Fundamental groups are highly non-commutative, but they have one algebraic operation, not two. This opens the perspective to try to perform deformations of these geometric objects not seen by algebraic geometry, using the fact that there are more maps, group homomorphisms and variations of those between topological groups in comparison to morphisms between commutative rings.

However, important associated diagrams are not commutative.

The main contribution of Mochizuki in the IUT theory for certain hyperbolic curves (e.g. an elliptic curve minus a point) is a new fundamental understanding of how to bound from above the deviation from commutativity of certain crucial diagrammes.

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On IUT on two pages

Algebraic geometry involves locally the correspondence between affine varieties and commutative rings.

Anabelian geometry for hyperbolic curves over number fields and other fields is a correspondence between these geometric objects and their arithmetic fundamental groups (or slightly more complicated objects).

Fundamental groups are highly non-commutative, but they have one algebraic operation, not two. This opens the perspective to try to perform deformations of these geometric objects not seen by algebraic geometry, using the fact that there are more maps, group homomorphisms and variations of those between topological groups in comparison to morphisms between commutative rings.

However, important associated diagrams are not commutative.

The main contribution of Mochizuki in the IUT theory for certain hyperbolic curves (e.g. an elliptic curve minus a point) is a new fundamental understanding of how to bound from above the deviation from commutativity of certain crucial diagrammes.

On IUT on two pages

IUT is a certain categorical monoidal geometry with few commutative diagrams but tools to measure their deviations from commutativity in certain situations which results in the new arithmetic deformation theory that is entirely unavailable via the standard arithmetic geometry.

IUT works with deformations of multiplication.

These deformations are not compatible with ring structure.

Deformations are coded in theta-links between theatres which are certain systems of categories associated to an elliptic curve over a number field.

Ring structures do not pass through theta-links.

Galois and fundamental groups (groups of symmetries of rings) do pass.

To restore certain rings from some groups that pass through a theta-link one uses anabelian geometry results about number fields and hyperbolic curves over them.

IUT is a *non-linear theory* which addresses such fundamental aspects as to which extent the multiplication and addition cannot be separated from one another.

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Similarities between IUT and quantum theory

There are many similarities.

1. An illustration of graphene hexagonal lattices during a talk at the Opening Event of the new Molecular Beam Epitaxy Facility for the growth of graphene and boron nitride layers has some similarities is interesting from the point of view of depicting such important aspects of IUT as symmetry, synchronisation, and the role of the number 6.

In IUT, multiplication and addition, which are related to geometric and arithmetic symmetries, play a central role. These two dimensions are reminiscent of the two parameters, one of which is related to electricity, the other to magnetism, employed in the study of layers of hexagonal lattices.

2. In mono-anabelian geometry and IUT one algorithmically reconstructs objects (fukugen) from étale fundamental groups.

IUT produces upper bounds on change of relevant data passing through the theta-link, using the action of étale fundamental groups which pass through the link unaffected.

Using appropriate group action on a flow of information, to control its loss, is useful in quantum computing.

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Similarities between IUT and quantum theory

3. IUT works with two types of topological monoid/group/ring structures: étale-like (coming from groups of symmetries) and frobenius-like (coming from 'ordered' objects) and then use various interactions and connections between these two structures. One of such connections is given by a generalised Kummer map.

From a certain perspective, the analogues of these two new math structures are waves and particles in quantum mechanics.

Interaction of frobenius-like and étale-like structures via the Kummer map in IUT may be sometimes viewed a little analogous to the relation between particles and waves in quantum mechanics.

4. The fact that in IUT it is only when one obtains a formal subquotient that forms a 'closed loop' then one may pass from subquotient to a set-theoretic subquotient by taking the log-volume is a little similar to a measurement of a quantum system with the wave function collapse.

5. One of the key issues for quantum algorithms is whether they can run in polynomial time, instead of exponential time. The aspect of reducing exponential to polynomial is crucial for IUT.

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